# Chapter 11

# Waves in lossless guides

In the preceding chapters we have seen that when the media are lossless the corresponding eigenvalue problems become, in most cases, symmetric. The symmetry of the operators substantially simplifies the investigation of their properties. Therefore we shall now turn to the guides filled with loss free media in order to establish useful relations for modes[1]. The analysis will be based, in principle, on the operator formalism developed in the preceding chapters. However, before applying this formalism to the investigation of modes in lossless guide we shall briefly present how classical methods can be used to achieve similar results.

## 11.1 Classical approach

A classical approach to the analysis of guided waves is based on direct manipulation of Maxwell's equations (Brown; 1966; Chorney et al.; 1967; Chorney and Penfield; 1971; Kogelnik and Weber; 1974) or application of the basic theorems of electromagnetics such as Lorentz equivalence principle (Collin; 1991; Rozzi et al.; 1996) or Poynting energy theorem (Chorney; 1961; Laxpati and Mittra; 1965) to the fields, varying according to $\exp(-j\beta z)$ factor. In order to give the reader an example of classical techniques we now derive two fundamental properties of propagating waves.

[1]It has to be noted that the Kramers-Kronig relations (Vassallo; 1980; Melrose and McPhedran; 1991) show that the loss free media are non-casual. In reality then, all physically realizable materials will dissipate energy. In the engineering practice, however, one most often uses low loss media. Hence, the assumption of loss free character of the wave guidance problem often is a quite realistic approximation and the results obtained are useful for practical purpose.

### 11.1.1 Energy balance for propagating waves

We begin with wave energy considerations valid for propagating waves in all types of waveguides. For this purpose we shall use the Poynting energy theorem for waveguides derived in Section 5.1.2. According to this theorem, the fields in a waveguide have to satisfy the relation

$$\frac{\partial}{\partial z}\int_{\Omega}(\vec{E}\times\vec{H}^*)\cdot\hat{z}d\Omega = j\omega\int_{\Omega}(\epsilon_0\vec{E}\cdot\overleftrightarrow{\epsilon}^*\cdot\vec{E}^* - \mu_0\vec{H}^*\cdot\overleftrightarrow{\mu}\cdot\vec{H})d\Omega \tag{11.1.1}$$

where $\Omega$ is the cross-section of the waveguide. Consider this theorem for the fields of a propagating mode. Since $\beta$ for such a mode is purely real, the term $(\vec{E}\times\vec{H}^*)$ does not depend on $z$ and hence the left hand side of (11.1.1) vanishes. This leads to the conclusion that for all propagating waves

$$\int_{\Omega}(\epsilon_0\vec{E}\cdot\overleftrightarrow{\epsilon}^*\cdot\vec{E}^* - \mu_0\vec{H}^*\cdot\overleftrightarrow{\mu}\cdot\vec{H})d\Omega = 0 \tag{11.1.2}$$

Since we assumed that the medium is lossless, the above equation can be written as

$$U_e = U_h \tag{11.1.3}$$

where $U_e$ and $U_h$ are the per-length electric and magnetic pseudo-energies, defined, by analogy to (5.1.15), as

$$U_e = \frac{1}{4}\int_{\Omega}\epsilon_0\vec{E}^*\cdot\overleftrightarrow{\epsilon}\cdot\vec{E}\,d\Omega \quad U_h = \frac{1}{4}\int_{\Omega}\mu_0\vec{H}^*\cdot\overleftrightarrow{\mu}\cdot\vec{H}\,d\Omega \tag{11.1.4}$$

### 11.1.2 Velocity of energy transport

As a second problem we shall investigate the velocity of energy transport. This velocity is one of several different characteristic velocities which can be associated with any type of wave motion. Among the most often used are the *phase velocity*

$$v_p = \frac{\omega}{\beta} \tag{11.1.5}$$

the *group velocity*

$$v_g = \frac{\partial}{\partial\beta}\omega \tag{11.1.6}$$

and the *energy-transport velocity*

$$v_e = \frac{P}{W_s} \tag{11.1.7}$$

where $W_s$ and $P$ are the density and flux density of the energy transport by the wave, averaged over the oscillation period and the wavelength.

The relations between these quantities have been investigated by Brillouin (1960) for simple cases but a thorough analysis of these concepts in the context of wave motion in an arbitrary system is due to Lighthill (1965). In this section we shall again use the Poynting theorem to find the relation between the group velocity and the velocity of energy transport. We cannot use the formula (11.1.1) directly because it applies to strictly monochromatic waves. In fact the guided wave propagation is a dispersive phenomenon so we should examine the Poynting theorem also for the weakly anharmonic fields. Let us assume that the propagation constant is a function of frequency and hence the field variation in the $z$ direction is described by the factor $e^{-(j\beta(\omega)z)}$. In Section 5.2.1 we showed that when the dispersion is present, equation (11.1.1) takes up the form

$$\begin{aligned}\frac{\partial}{\partial z}\int_\Omega \left(\frac{\partial}{\partial \omega}\vec{E}\times\vec{H}^* + \vec{E}^*\times\frac{\partial}{\partial\omega}\vec{H}\right)\cdot\hat{z}d\Omega = \\ -j\int_\Omega \left[\mu_0\vec{H}^*\cdot\frac{\partial}{\partial\omega}\left(\omega\overleftrightarrow{\mu}\right)\vec{H} + \epsilon_0\vec{E}^*\cdot\frac{\partial}{\partial\omega}\left(\omega\overleftrightarrow{\epsilon}\right)\vec{E}\right]\,d\Omega\end{aligned} \tag{11.1.8}$$

For the assumed variation of the field in the $z$ direction the left hand side term in (11.1.8) becomes

$$-j\frac{\partial}{\partial\omega}\beta\frac{\partial}{\partial z}\int_\Omega z\left(\vec{E}\times\vec{H}^* + \vec{E}^*\times\vec{H}\right)\cdot\hat{z}\,d\Omega \tag{11.1.9}$$

whence we immediately get for the propagating waves ($\beta$ real)

$$\frac{\partial}{\partial\omega}\beta = \frac{\int_\Omega \left[\mu_0\vec{H}^*\cdot\frac{\partial}{\partial\omega}\left(\omega\overleftrightarrow{\mu}\right)\vec{H} + \epsilon_0\vec{E}^*\cdot\frac{\partial}{\partial\omega}\left(\omega\overleftrightarrow{\epsilon}\right)\vec{E}\right]\,d\Omega}{\int_\Omega\left(\vec{E}\times\vec{H}^* + \vec{E}^*\times\vec{H}\right)\cdot\hat{z}\,d\Omega} \tag{11.1.10}$$

Since the derivative of the propagation constant is the inverse of the *group velocity*, we have

$$v_g = \frac{P}{W_s} = \frac{P}{W_e + W_h} \tag{11.1.11}$$

with

$$P = \frac{1}{4}\int_\Omega\left(\vec{E}\times\vec{H}^* + \vec{E}^*\times\vec{H}\right)\cdot\hat{z}\,d\Omega = \frac{1}{2}\mathcal{R}e\int_\Omega\left(\vec{E}\times\vec{H}^*\right)\cdot\hat{z}\,d\Omega \tag{11.1.12}$$

where we used the fact that $2\mathcal{R}e\{A\} = A + A^*$ and defined $W_e$, $W_h$ as the per length true average electric and magnetic field energies

$$\begin{aligned}W_e &= \frac{\epsilon_0}{4}\int_\Omega\left[\vec{E}^*\frac{\partial}{\partial\omega}\left(\omega\overleftrightarrow{\epsilon}\right)\vec{E}\right]\,d\Omega \\ W_h &= \frac{\mu_0}{4}\int_\Omega\left[\vec{H}^*\frac{\partial}{\partial\omega}\left(\omega\overleftrightarrow{\mu}\right)\vec{H}\right]\,d\Omega\end{aligned} \tag{11.1.13}$$

In $P$ we recognize the expression for average power and hence the group velocity is the rate at with which a mode transports energy along the guide.

$$v_g = v_e \tag{11.1.14}$$

## 11.2 Operator formalism

We now pass on to the presentation of a more versatile approach in which we shall directly use the results of the detailed analysis of operators describing wave propagation.

### 11.2.1 Group velocity - revisited

As an example of the versatility of the operator formalism, it is illustrative to show that the expression for the group velocity is derivable in a much simpler and elegant way, using only the properties of the operators and variational calculus (Vassallo; 1985). In Section 4.1 it was shown that the variational expression for *real* eigenvalues of a symmetric generalized eigenvalue problem is

$$\lambda = \frac{(\mathbf{A}x, x)}{(\mathbf{M}x, x)} \tag{11.2.1}$$

If the medium is lossless and we are only interested in finding the propagating waves, a necessary condition for the application of the expression (11.2.1) is fulfilled and we may apply the variational principle to find the group velocity. Substituting (7.1.2) and (7.1.3) for $\mathbf{A}$ and $\mathbf{M}$ in (11.2.1) we get the variational formula for modes propagating in an arbitrary lossless waveguide

$$\beta = \frac{\displaystyle\int_\Omega \left[\mu_0 \vec{H}^* \left(\omega \overleftrightarrow{\mu}\right) \vec{H} + \epsilon_0 \vec{E}^* \left(\omega \overleftrightarrow{\epsilon}\right) \vec{E} - j\vec{H}^* \cdot \nabla_t \times \vec{E} + j\vec{E}^* \cdot \nabla_t \times \vec{H}\right] \, d\Omega}{\displaystyle\int_\Omega \left(\vec{E} \times \vec{H}^* + \vec{E}^* \times \vec{H}\right) \cdot \hat{z} d\Omega} \tag{11.2.2}$$

This expression may be written in the form

$$\beta = \mathbf{F}_0(\omega, \underline{\Phi}) \tag{11.2.3}$$

Calculation of the derivative of $\beta$ with respect to $\omega$ is now straightforward. The complete differential of $\mathbf{F}_0$ is

$$\delta\beta = \frac{\partial}{\partial\omega}\mathbf{F}_0|_{\underline{\Phi}=\text{const}}\delta\omega + \delta\mathbf{F}_0|_{\omega=\text{const}} \tag{11.2.4}$$

$\mathbf{F}_0$ being stationary with respect to variation of $\underline{\Phi}$ implies vanishing of the second term, and consequently

$$\delta\beta = \frac{\partial}{\partial\omega}\mathbf{F}_0|_{\underline{\Phi}=\text{const}}\delta\omega \tag{11.2.5}$$

Whence

$$\frac{\partial}{\partial\omega}\beta = \frac{\partial}{\partial\omega}\mathbf{F}_0|_{\underline{\Phi}=\text{const}} \tag{11.2.6}$$

Application of the formula derived above to the expression (11.2.2) yields (Vassallo; 1985)

$$\frac{\partial}{\partial\omega}\beta = \frac{1}{v_g} = \frac{\int_\Omega \left[\mu_0 \vec{H}^* \cdot \frac{\partial}{\partial\omega}\left(\omega\overleftrightarrow{\mu}\right)\cdot\vec{H} + \epsilon_0\vec{E}^*\cdot\frac{\partial}{\partial\omega}\left(\omega\overleftrightarrow{\epsilon}\right)\cdot\vec{E}\right]\, d\Omega}{\int_\Omega \left(\vec{E}\times\vec{H}^* + \vec{E}^*\times\vec{H}\right)\cdot\hat{z}\, d\Omega} \tag{11.2.7}$$

or

$$v_g = \frac{2\mathcal{R}e\int_\Omega \left(\vec{E}\times\vec{H}^*\right)\cdot\hat{z}d\Omega}{\int_\Omega \left[\mu_0 \vec{H}^* \cdot \frac{\partial}{\partial\omega}\left(\omega\overleftrightarrow{\mu}\right)\cdot\vec{H} + \epsilon_0\vec{E}^*\cdot\frac{\partial}{\partial\omega}\left(\omega\overleftrightarrow{\epsilon}\right)\cdot\vec{E}\right]\, d\Omega} \tag{11.2.8}$$

In conclusion we may state that the result obtained is identical to (11.1.10) but was derived without resorting to the cumbersome analysis of Maxwell's equations under anharmonic excitation. The variational technique is more general and can be applied to other cases.

As a final comment we observe that the group velocity loses its physical meaning if $\omega$ or $\beta$ becomes complex. Also the energy transport velocity then needs a different interpretation since $W$ and $P$ become functions of time and position. Silbergleit and Kopilevich (1980b,a) suggest that the correct definition of the energy transport velocity in such cases is given by the ratio of the maximum amplitude of the average energy flux to the maximum amplitude of the average energy flux density.

### 11.2.2 Energy, power and momentum expressions

We showed that an expression for the group velocity can be derived using the operator formalism. Also, the energy balance formula derived from the Poynting theorem can easily be deduced from the analysis of the quadratic forms created by taking the inner product of generalized eigenvalue problems with suitable testing functions. This technique, not necessarily limited to lossless media, was proven in Sections 7.4 and 7.5 to give the power-energy and momentum-energy expressions for modes. We shall now investigate these expressions under a steady harmonic state ($\mathcal{I}m\,\omega = 0$) in some detail, assuming that there is no loss in the system. If the medium is lossless then the electric and magnetic pseudo-energy densities $\hat{U}_e^{ii}$

and $\hat{U}_h^{ii}$ are real, so we shall drop the carets denoting complex quantities over these symbols. On the other hand, the longitudinal and transverse pseudo-energies are in general complex, so the carets have to be preserved.

In Section 2.2.1 we divided modes in loss free guides into three broad categories, which classify a mode as a propagating, cutoff (evanescent) or complex (semi-evanescent) wave depending on whether the propagation constant is purely real, imaginary or complex. It is sensible to investigate the properties of these three categories separately.

## 11.3 Propagating waves

When the wave is propagating $\mathcal{I}m\,\beta = 0$ expressions (7.4.16), (7.4.17), (7.5.9) and (7.5.10) become[2]

$$\beta(P + jQ) = 2\omega(\hat{U}_{et} - \hat{U}_{hz}) \tag{11.3.1}$$

$$\beta(P + jQ) = -2\omega(\hat{U}_{ez} - \hat{U}_{ht}) \tag{11.3.2}$$

$$\omega(\mathcal{G} - j\mathcal{F}) = 2\beta(\hat{U}_{et} + \hat{U}_{ez}) \tag{11.3.3}$$

$$\omega(\mathcal{G} - j\mathcal{F}) = 2\beta(\hat{U}_{ht} + \hat{U}_{hz}) \tag{11.3.4}$$

By taking a linear combination of the relevant expressions the following corollaries ensue

$$P = \frac{\omega}{\beta}\mathcal{R}e\,(\hat{U}_t - \hat{U}_z) \tag{11.3.5}$$

$$Q = \frac{\omega}{\beta}\mathcal{I}m\,(\hat{U}_t - \hat{U}_z) \tag{11.3.6}$$

$$U_e = U_h \tag{11.3.7}$$

$$\mathcal{G} = \frac{\beta}{\omega}(U_e + U_h) \tag{11.3.8}$$

$$\mathcal{F} = 0 \tag{11.3.9}$$

From (11.3.7) we also find that the following identity must be true

$$2(\hat{U}_{ht} - \hat{U}_{ez}) = 2(\hat{U}_{et} - \hat{U}_{hz}) = (\hat{U}_t - \hat{U}_z) \tag{11.3.10}$$

For strictly bidirectional guides the longitudinal and transverse pseudo-energies are purely real. Consequently (11.3.5), (11.3.6) and (11.3.10) assume a simpler form[3]

$$P = \frac{\omega}{\beta}(U_t - U_z) \tag{11.3.11}$$

[2] Since in most cases expressions used are valid for the fields of a single wave, we have dropped the sub- or superscripts $ij$ indicating the interaction between modes $i$ and $j$. These indices will be restored if needed.

[3] Obviously expressions (11.3.7)-(11.3.9) still hold.

$$Q = 0 \tag{11.3.12}$$

$$2(U_{ht} - U_{ez}) = 2(U_{et} - U_{hz}) = (U_t - U_z) \tag{11.3.13}$$

with $U_t$ and $U_z$, the per-length pseudo-energies defined as

$$U_t = U_{et} + U_{ht} = \frac{1}{4}\int_\Omega \left(\epsilon_0 \vec{E}_{ti}^{\,*} \cdot \underline{\underline{\epsilon}}_t \vec{E}_{ti} + \mu_0 \vec{H}_{ti}^{\,*} \cdot \underline{\underline{\mu}}_t \vec{H}_{ti}\right) d\Omega \tag{11.3.14}$$

and

$$U_z = U_{hz} + U_{ez} = \frac{1}{4}\int_\Omega \left(\mu_0 \mu_{zz} |\vec{H}_{zi}|^2 + \epsilon_0 \epsilon_{zz} |\vec{E}_{zi}|^2\right) d\Omega \tag{11.3.15}$$

Combining now (11.3.8) and (11.3.11) one gets

$$P\mathcal{G} = U_t^2 - U_z^2 \tag{11.3.16}$$

The above expressions should also be completed with the results obtained in the previous sections. We have just found out that in general

$$P = v_g W_s \tag{11.3.17}$$

Additionally, in Section 8.2.2 we established, considering the properties of the complementary modes, that for strictly bidirectional guides loaded with reciprocal media the transverse and longitudinal field components are, respectively, real and imaginary

$$\vec{E}_t = \vec{E}_t^{\,*}, \vec{E}_z = -\vec{E}_z^{\,*},\ \vec{H}_t = \vec{H}_t^{\,*},\ \vec{H}_z = -\vec{H}_z^{\,*} \tag{11.3.18}$$

We have now collected a set of basic relations valid for propagating waves. Based on these expressions many interesting properties of waves can be found.

### 11.3.1 Boundedness of group velocity

We have interpreted the group velocity as the rate at which the energy propagates along the guide. According to relativistic physics, the absolute value of energy transport velocity is bounded from above. Let us see if this can be proved using the operator formalism. The boundedness of $v_g$ requires the boundedness of the numerator and nonzero value of the denominator in (11.2.8). The numerator is bounded because it expresses the average power transmitted by the mode and this quantity is obviously finite. In Section 5.3 it was shown that the constitutive parameter tensors in a passive lossless medium must lead to the following positive hermitian forms

$$\vec{A}^* \cdot \frac{\partial}{\partial \omega}(\omega \overleftrightarrow{\epsilon}) \cdot \vec{A} > 0 \qquad \text{and} \qquad \vec{A}^* \cdot \frac{\partial}{\partial \omega}(\omega \overleftrightarrow{\mu}) \cdot \vec{A} > 0 \tag{11.3.19}$$

As a result the denominator is always positive and, accordingly, we may state that the group velocity is finite.

From the positive definiteness of the hermitian forms in the denominator it also follows that the sign change of the group velocity will always be associated with the sign change of the expression in the numerator. The integrand in the numerator is the $z$-component of the Poynting vector. Accordingly, the integration over the guide cross-section gives the net average value of the power transported in $+z$ direction. The negative values of the group velocity indicate then that the energy transfer takes place in the $-z$ direction. It has also to be noted that the group velocity can become zero, indicating that energy is not transported by the guide. This is possible when $P = 0$ or when the terms in the denominator are singular.

### 11.3.2 Forward and backward waves

Introducing the mode nomenclature in Section 2.2.1 we broadly divided the propagating waves into the forward and backward waves. The distinction between these classes was made based on the slope of the $\mathcal{R}e\,\{\beta(\omega)\}$ curves. The modes with the positive slope were termed forward waves and modes with the negative slope were called backward waves. In view of the discussions presented in the previous section, this designation now becomes clear. The slope of the dispersion curves is the inverse of the group velocity and hence its negative values are indeed associated with the energy transfer in the $-z$ direction. At this point it should be noted that so far, for instance, when discussing bidirectionality, we used the term 'forward' and 'backward' rather loosely, without actually investigating the direction of the energy transfer. 'Forward' and 'backward' (or 'reverse') waves were named according to the sign of the expression in the phase term. In fact the phase term $\exp\{-j\beta z\}$ indicates only that the phase delay increases when $z$ increases. To avoid ambiguity one should therefore use terms 'direct' and 'complementary' for distinguishing the modes with $\pm\beta$[4]. In Section 8.2.1 we showed that the relation between fields of direct and complementary modes is known for all propagating waves provided the guide contains only lossless reciprocal medium. If this is the case then in order to obtain fields of the complementary mode one has to take the complex conjugate of the fields of the direct mode and reverse the sign of the magnetic field (cf. eq. (8.2.2)).

Returning to the forward and backward waves, it is readily inferred from relation

[4]Some authors (Rozzi et al.; 1996) use the names 'progressive' and 'regressive' for the same purpose.

(11.3.5) that the following inequalities must take place

$$\mathcal{R}e\,\hat{U}_t > \mathcal{R}e\,\hat{U}_z \qquad \text{for forward waves} \tag{11.3.20}$$

and

$$\mathcal{R}e\,\hat{U}_t < \mathcal{R}e\,\hat{U}_z \qquad \text{for backward waves} \tag{11.3.21}$$

The relation between the group velocity and the slope of the $\mathcal{R}e\,\{\beta(\omega)\}$ curves gives additional information about the admissible behaviour of the dispersion characteristics. Since the group velocity is bounded from above, the slope of the $\mathcal{R}e\,\{\beta(\omega)\}$ curves for $\mathcal{I}m\,\{\beta(\omega)\} = 0$ has to be nonzero. In other words the correct physical behaviour of the field precludes the existence of inflection points on the dispersion characteristics for all modes above cutoff. As we shall see in the subsequent sections this is not the case for the $\mathcal{I}m\,\{\beta(\omega)\}$ curves where such inflection points are possible.

### 11.3.3 Phase and group velocities

The *phase velocity* is one of the characteristic velocities associated with wave motion. According to (11.1.5) the phase velocity is simply the ratio of the angular frequency and the propagation constant. For a plane wave propagation in a homogeneous isotropic medium the phase and group velocities are equal (Johnson; 1965), (Vassallo; 1980). The dispersive propagation implies that these two quantities no longer have identical values. It is interesting to investigate the general relation between these two velocities.

Formula (11.3.5) is useful for expressing the phase velocity $v_p$ in terms of the pseudo-energies. Transforming (11.3.5) we get

$$v_p = \frac{\omega}{\beta} = \frac{P}{\mathcal{R}e\,(\hat{U}_t - \hat{U}_z)} \tag{11.3.22}$$

By virtue of (11.3.10) formula (11.3.22) is equivalent to the following expressions

$$v_p = \frac{\omega}{\beta} = \frac{P}{2\mathcal{R}e\,(\hat{U}_{ht} - \hat{U}_{ez})} = \frac{P}{2\mathcal{R}e\,(\hat{U}_{et} - \hat{U}_{hz})} \tag{11.3.23}$$

We may now find the relation between the phase and group velocity. We showed that the group velocity is given by

$$v_g = \frac{P}{W_s} = \frac{P}{W_e + W_h} \tag{11.3.24}$$

where $W_e$ and $W_h$ are the true per-length electric and magnetic energies defined in (11.1.13). Dividing both sides of (11.3.22) by the corresponding expressions from (11.3.24) we get the formula relating the phase and group velocity

$$\frac{v_p}{v_g} = \frac{W_s}{\mathcal{R}e\,(\hat{U}_t - \hat{U}_z)} \tag{11.3.25}$$

or its equivalents in the form

$$\frac{v_p}{v_g} = \frac{W_s}{2\mathcal{R}e\,(\hat{U}_{ht} - \hat{U}_{ez})} = \frac{W_s}{2\mathcal{R}e\,(\hat{U}_{et} - \hat{U}_{hz})} \tag{11.3.26}$$

**Nondispersive media**

For nondispersive media formula (11.3.25) can be further simplified. When the constitutive parameters do not depend on frequency then $W_e = U_e$ and $W_h = U_h$, whence (11.3.25) becomes

$$\frac{v_p}{v_g} = \frac{U_t + U_e}{\mathcal{R}e\,(\hat{U}_t - \hat{U}_z)} = \frac{\hat{U}_t + \hat{U}_z}{\mathcal{R}e\,(\hat{U}_t - \hat{U}_z)} \tag{11.3.27}$$

The above expression allows us to compare the magnitudes of the group and phase velocities

$$|v_p| = |v_g|\frac{\hat{U}_t + \hat{U}_z}{|\mathcal{R}e\,(\hat{U}_t - \hat{U}_z)|} \geq |v_g| \tag{11.3.28}$$

We see that the magnitude of phase velocity of hybrid modes always exceeds the magnitude of group velocity. These two quantities are equal only when $\hat{U}_z = 0$ and $\hat{U}_t = \mathcal{R}e\,\hat{U}_t$. The former situation takes place for the $TEM$ modes and the latter in strictly bidirectional guides.

On the other hand, since $v_p = \omega/\beta$, the following inequality holds

$$\omega \geq |v_g||\beta| \tag{11.3.29}$$

$v_g$ was proven to be finite so $\beta$ must also be finite for finite $\omega$. This proves that spectra of the eigenvalue problems describing guided wave propagation in closed structures are bounded from above on the real axis.

An important identity can be found by considering the relation (11.3.24) in a nondispersive medium together with (11.3.8)

$$v_g = \frac{P}{W_s} = \frac{P}{U_e + U_h} \tag{11.3.30}$$

From (11.3.8)

$$v_p\mathcal{G} = (U_e + U_h) = W_s \tag{11.3.31}$$

substituting this result into (11.3.30) one gets

$$v_g v_p = \frac{P}{\mathcal{G}} \tag{11.3.32}$$

in explicit form, the above formula reads

$$v_g v_p = \frac{\mathcal{R}e \int_\Omega \left(\vec{E}_t \times \vec{H}_t{}^*\right) \cdot \hat{z} d\Omega}{\mathcal{R}e \int_\Omega \left(\vec{D}_t \times \vec{B}_t{}^*\right) \cdot \hat{z} d\Omega} \tag{11.3.33}$$

In a similar vein, (11.3.27) can be substituted into (11.3.31) to give

$$v_g \mathcal{G} = \mathcal{R}e\,(\hat{U}_t - \hat{U}_z) \tag{11.3.34}$$

Formula (11.3.32) relates the time-averages of the $z$ components of Poynting and electromagnetic momentum vectors to the group and phase velocities. This formula, first derived for layered dielectric guide by Semenov (1963), is known in the Russian literature as the partial-power law. The same expression, this time for a more general case of guides containing arbitrary inhomogeneous isotropic media was later obtained by Brown (1966) and is sometimes referred to as Brown's identity (Vassallo; 1985). An example of the application of Brown's identity is the investigation of the group velocity in multi-mode optical fibre (Gloge; 1975). The extension of this identity to the case of arbitrary anisotropic media was made by Haus and Kogelnik (1976), who used classical methods. It has to be mentioned that while relation (11.3.33) was obtained under the assumption that the guide is loss free, a generalization to the waveguides with lossy media is also possible (Kuester; 1984).

### 11.3.4 Guides loaded with isotropic dielectric

We shall now apply the partial-power law to find the bounds on the group and phase velocities in guides containing only isotropic media. When the medium is isotropic the partial-power law becomes

$$\frac{1}{v_p v_g} = \frac{\int_\Omega \epsilon\mu \left(\vec{E} \times \vec{H}^*\right) \cdot \hat{z} d\Omega}{c^2 \int_\Omega \left(\vec{E} \times \vec{H}^*\right) \cdot \hat{z} d\Omega} \tag{11.3.35}$$

with $c$ denoting the velocity of light in the free space. It is now easy to give the following estimates

$$\frac{(\epsilon\mu)_{\min}}{c^2} \leq \frac{\int_\Omega \epsilon\mu \left(\vec{E} \times \vec{H}^*\right) \cdot \hat{z}\, d\Omega}{c^2 \int_\Omega \left(\vec{E} \times \vec{H}^*\right) \cdot \hat{z}\, d\Omega} \leq \frac{(\epsilon\mu)_{\max}}{c^2} \tag{11.3.36}$$

where

$$(\epsilon\mu)_{\min} = \min_{\Omega} \epsilon\mu \qquad (\epsilon\mu)_{\max} = \max_{\Omega} \epsilon\mu \tag{11.3.37}$$

It follows from inequality (11.3.28) that

$$v_p^2 \geq \frac{c^2}{(\epsilon\mu)_{\max}} \quad \text{and} \quad v_g^2 \leq \frac{c^2}{(\epsilon\mu)_{\min}} \tag{11.3.38}$$

The interpretation of the second of the above inequalities is that the energy in a waveguide is always transported with the velocity which is lower than the velocity of the plane wave in a homogeneous medium with constitutive parameters equal to the minimal value of product of permittivity and permeability. By taking the reciprocal we get the bound on the minimum slope of the dispersion curves for propagating waves.

Finally, when the medium is homogeneous (11.3.35) becomes

$$v_p v_g = \frac{c^2}{\epsilon\mu} \tag{11.3.39}$$

This result is well known from the theory of $TE$ and $TM$ modes (Collin; 1991; Johnson; 1965).

### 11.3.5 Physical interpretation - energy and momentum conservation

Following Haus and Kogelnik (1976) we shall now give a physical interpretation of some of the relations derived so far. Throughout this book we present the classical wave approach to electromagnetic phenomena. However, in the quantum-mechanical picture guided electromagnetic waves can alternatively be understood as wave packets consisting of a certain number of quasiparticles per unit length of the guide. In such a quantum-mechanical interpretation the energy stored in the wave packed per unit length is

$$W_s = N\frac{h}{2\pi}\omega = N\hbar\omega \tag{11.3.40}$$

where $N$ is the linear density of quasiparticles and $h = 2\pi\hbar$ is the Planck constant. Likewise the momentum per unit length of the guide is

$$\mathcal{G} = N\hbar\beta \tag{11.3.41}$$

The wave packet moves with group velocity $v_g$ and hence the rates of the energy transport $P$ and the momentum flow $\mathcal{T}$ are given by

$$P = v_g N\hbar\omega \quad \text{and} \quad \mathcal{T} = v_g N\hbar\beta \tag{11.3.42}$$

Manipulating the above expressions one gets

$$P = v_g W_s \tag{11.3.43}$$

$$\mathcal{T} = v_g \mathcal{G} \tag{11.3.44}$$

$$W_s = \frac{\omega}{\beta}\mathcal{G} = v_p \mathcal{G} \tag{11.3.45}$$

$$P = \frac{\omega}{\beta}\mathcal{T} = v_p \mathcal{T} \tag{11.3.46}$$

From (11.3.44) and (11.3.46) it also follows that

$$P = v_g v_p \mathcal{G} \tag{11.3.47}$$

In the above five relations one can easily recognize some of the expressions derived earlier in this chapter. In particular (11.3.43) and (11.3.45) correspond to, respectively, (11.3.17) and (11.3.22). If one denotes

$$\mathcal{T} = \mathcal{R}e\,(\hat{U}_t - \hat{U}_z) \tag{11.3.48}$$

then a similar correspondence can be established between (11.3.44) and expression (11.3.31), or between (11.3.46) and (11.3.34). Finally in (11.3.47) we recognize the partial-power law (Brown's identity) (11.3.32). It is now seen that the quantum-mechanical picture of wave phenomena yields the same results as the classical wave approach. Furthermore one may use this model to infer additional properties which are not easily seen directly from Maxwell's equations. For instance, (11.3.43) and (11.3.44) are the quantum-mechanical forms of energy and momentum conservation. The differential forms of these relations, for a wave packet travelling in the $z$ direction, are

$$\frac{\partial}{\partial t} W_s + \frac{\partial}{\partial z} P = 0 \tag{11.3.49}$$

for the energy conservation theorem, and

$$\frac{\partial}{\partial t}\mathcal{G} + \frac{\partial}{\partial z}\mathcal{T} = 0 \tag{11.3.50}$$

for the electromagnetic momentum conservation theorem.

The three dimensional electromagnetic energy and momentum conservation theorems are readily obtained directly from Maxwell's equations (Stratton; 1941; Kong; 1990), but the derivation of one dimensional differential forms of conservation theorems is much more involved[5]. Gureev (1990) showed that this derivation,

[5]Expression (5.2.38) derived in Chapter 5 using the technique of Tonning (1960), is the differential form of the energy conservation theorem. The derivation of the momentum conservation theorem can be carried out a similar manner (Haus and Kogelnik; 1976).

however, becomes much simpler if we restrict ourselves to strictly bidirectional media. In Section 9.2 we established that the wave propagation in strictly bidirectional guides is described by two quadratic pencils. The differential forms of pencils $\mathbf{X}$ and $\mathbf{Y}$ are

$$\frac{\partial^2}{\partial t^2}\mathbf{X}_{\omega^2}\underline{\Psi} + \frac{\partial^2}{\partial t \partial z}\mathbf{X}_{\omega\beta}\underline{\Psi} + \mathbf{X}_0\underline{\Psi} = 0 \tag{11.3.51}$$

$$\frac{\partial^2}{\partial z^2}\mathbf{Y}_{\beta^2}\underline{\Psi} + \frac{\partial^2}{\partial t \partial z}\mathbf{Y}_{\omega\beta}\underline{\Psi} + \mathbf{Y}_0\underline{\Psi} = 0 \tag{11.3.52}$$

If field $\underline{\Psi}$ satisfies the principle of causality then taking the inner product of (11.3.51) with $\underline{\Psi}$, integrated with respect to $t$ from 0 to $t' = t$, gives

$$\frac{\partial}{\partial t}(U_e + U_h) + \frac{\partial}{\partial z}P = 0 \tag{11.3.53}$$

Similarly for field $\underline{\Psi}$ satisfying the radiation condition, the inner product of (11.3.52) with $\underline{\Psi}$, integrated with respect to $z$ from 0 to $z' = z$, yields

$$\frac{\partial}{\partial t}\mathcal{G} + \frac{\partial}{\partial z}(U_t - U_z) = \frac{\partial}{\partial t}\mathcal{G} + \frac{\partial}{\partial z}\mathcal{T} = 0 \tag{11.3.54}$$

where (Gureev; 1990)

$$\begin{aligned} U_z &= \frac{1}{4}\Big(\mathbf{X}_0 \int_0^{t'} \underline{\Psi}\, dt, \int_0^{t'} \underline{\Psi}\, dt\Big)_{|t'=t} \\ &= -\frac{1}{4}\Big(\mathbf{Y}_0 \int_{\pm\infty}^{z'} \underline{\Psi}\, dz, \int_{\pm\infty}^{z'} \underline{\Psi}\, dz\Big)_{|z'=z} \\ &= \frac{1}{4}\int_\Omega \Big(\mu_0\mu_{zz}|\vec{H}_z|^2 + \epsilon_0\epsilon_{zz}|\vec{E}_z|^2\Big)\, d\Omega \end{aligned} \tag{11.3.55}$$

We now see that pencils $\mathbf{X}$ and $\mathbf{Y}$ in fact express the conservative properties of electromagnetic fields in terms of energy and electromagnetic momentum, respectively.

It can be noted that so far we have been interpreting the momentum flow $\mathcal{T}$ in terms of difference of transverse and longitudinal pseudo-energies. In fact a direct inspection shows that it is related to the $zz$ component of the Maxwell stress tensor (Kong; 1990)

$$\overleftrightarrow{\mathcal{T}} = \frac{1}{2}\mathcal{R}e\,[\vec{D}^*\vec{E} + \vec{B}^*\vec{H} - \frac{1}{2}\mathbf{I}(\vec{D}^*\cdot\vec{E} + \vec{B}^*\cdot\vec{H})] \tag{11.3.56}$$

where $\mathbf{I}$ is the identity tensor, $\vec{D}^*\vec{E}$ and $\vec{B}^*\vec{H}$ are dyads (VanBladel; 1964) and ordinary vector multiplications are denoted by dots. Evaluation of the $zz$ component of $\overleftrightarrow{\mathcal{T}}$ gives, indeed,

$$\overleftrightarrow{\mathcal{T}}_{zz} = \mathcal{R}e\,(\hat{U}_t - \hat{U}_z) \tag{11.3.57}$$

At this point we stop our investigation of physical interpretation of the formulae derived for guided modes. For the in depth discussion of subtleties, especially regarding the electromagnetic momentum, the paper of Haus and Kogelnik (1976) can be particularly recommended.

## 11.4 Cutoff waves

A second type of waves which can be expected in lossless structures are the modes with purely imaginary propagation constants which have been termed the *cutoff* or *evanescent* waves. To start the analysis of the cutoff waves we have to establish first the conditions for the existence of purely imaginary values of a propagation constant.

To investigate this problem we shall use the fact that the spectrum of a transposed problem is the complex conjugate of the spectrum of the original problem. As noted in Section 7.2.1 both operators **A** and **M** are symmetric if the guide is lossless and hence the solutions of the transposed problem are identical to the solutions of the original problem. In other words the solution corresponding to the complex conjugate of each complex eigenvalue must be the solution of the problem[6]. Consequently if $\beta$ for cutoff waves is purely imaginary then its conjugate is $-\beta$ and a mode with such a propagation constant must also be supported by the guide. Let us suppose that the existence of the $\pm\beta$ pairs for imaginary $\beta$ entails the existence of such pairs for real $\beta$. Complementary pairs with propagation constants $\pm\beta$ exist in bidirectional structures. This allows us to conclude that in order to support waves with purely imaginary propagation constants (cutoff waves) it is necessary for the waveguide to be bidirectional. All bidirectional guides have been identified in Section 8.2 and we may therefore infer that the cutoff waves exist in:

1. Waveguides loaded with lossless nongyrotropic media.
2. Waveguides loaded with lossless but possibly gyrotropic media whose constitutive parameters or cross-sectional geometries are invariant to the $z$ axis reversal obtained via reflection, inversion or rotation operations discussed in Section 8.2.

No other waveguiding structures support purely cutoff (evanescent) waves.

### 11.4.1 Power and momentum identities for cutoff waves

We have shown that propagating waves in all types of waveguides do not store the energy in the form of reactive power ($Q = 0$) and must feature the average energy

[6]Complex eigenvalues can occur if both operators are indefinite (see Section 3.2.1).

balance expressed as $U_e = U_h$. We also showed that many power-energy-momentum identities exist. It is interesting to examine if similar formulae can be found for the cutoff waves. One such identity can immediately be found from the condition of the existence of nonreal eigenvalues in symmetric generalized eigenproblems. It was shown in Section 3.2.1 that a nonreal eigenvalue in a symmetric generalized eigenproblem implies $(\mathbf{M}x, x) = (\mathbf{A}x, x) = 0$ (cf. eq. (3.2.15)). We also showed that the first of the forms is proportional to the power transported along the guide, which is expressed by the real part of the Poynting vector. Accordingly, we may state that

$$P = 0 \tag{11.4.1}$$

for all evanescent waves. Obviously one may also endow $(\mathbf{A}x, x) = 0$ with a physical interpretation but it is also worthwhile to investigate other forms in order to arrive at a larger set of identities for cutoff waves. To this end we shall again use expressions (7.4.16), (7.4.17), (7.5.9) and (7.5.10), which under the assumptions $\beta = \mathcal{R}e\,\beta - j\mathcal{I}m\,\beta$ and $\mathcal{R}e\,\beta = 0$, appear in the following form

$$(\mathcal{I}m\,\beta)(P + jQ) = -2j\omega(\hat{U}_{et} - \hat{U}_{hz}) \tag{11.4.2}$$

$$(\mathcal{I}m\,\beta)(P + jQ) = -2j\omega(\hat{U}_{ez} - \hat{U}_{ht}) \tag{11.4.3}$$

and

$$\omega(\mathcal{G} - j\mathcal{F}) = -2j(\mathcal{I}m\,\beta)(\hat{U}_{et} - \hat{U}_{ez}) \tag{11.4.4}$$

$$\omega(\mathcal{G} - j\mathcal{F}) = -2j(\mathcal{I}m\,\beta)(\hat{U}_{hz} - \hat{U}_{ht}) \tag{11.4.5}$$

The imaginary part of $\beta$ is the attenuation constant and is customarily denoted in the literature using the symbol $\alpha$. With this designation the following corollaries immediately ensue

$$\hat{U}_{hz} - \hat{U}_{ht} = \hat{U}_{et} - \hat{U}_{ez} \tag{11.4.6}$$

$$Q = \frac{\omega}{\alpha}(U_h - U_e) \tag{11.4.7}$$

Transforming the last expression gives

$$\hat{U}_t = \hat{U}_z \tag{11.4.8}$$

from which we get

$$U_e - U_h = 2(\hat{U}_{ez} - \hat{U}_{ht}) = 2(\hat{U}_{et} - \hat{U}_{hz}) \tag{11.4.9}$$

Since per-length densities of electric and magnetic pseudo-energies $U_e$ and $U_h$ are real then the following must also be true

$$\mathcal{I}m\,(\hat{U}_{ez} - \hat{U}_{ht}) = \mathcal{I}m\,(\hat{U}_{et} - \hat{U}_{hz}) = 0 \tag{11.4.10}$$

With these identities we obtain an alternative form of the expression for the reactive power

$$Q = \frac{2\omega}{\alpha}(\hat{U}_{ht} - \hat{U}_{ez}) = \frac{2\omega}{\alpha}(\hat{U}_{hz} - \hat{U}_{et}) \tag{11.4.11}$$

So far we have obtained only expressions involving power and energy. For the momentum, we find by adding (11.4.4) to (11.4.5) that

$$\mathcal{F} = \frac{\alpha}{\omega}\mathcal{R}e\,(\hat{U}_{et} - \hat{U}_{ez} - \hat{U}_{ht} + \hat{U}_{hz}) \tag{11.4.12}$$

$$\mathcal{G} = \frac{\alpha}{\omega}\mathcal{I}m\,(\hat{U}_{et} - \hat{U}_{ez} - \hat{U}_{ht} + \hat{U}_{hz}) \tag{11.4.13}$$

The above set of identities, applicable to all categories of guides supporting cutoff waves, can be complemented with the result which is valid only for reciprocal media. For fields in a lossless reciprocal medium it has been found out (see Section 8.2.1) that fields below cutoff are either purely imaginary or purely real

$$\mathcal{I}m\,\vec{E} = \mathcal{R}e\,\vec{H} = 0 \quad \text{when } \mathcal{R}e\,\beta = 0 \text{ and } \overleftrightarrow{\epsilon} = \overleftrightarrow{\epsilon}^*,\ \overleftrightarrow{\mu} = \overleftrightarrow{\mu}^* \tag{11.4.14}$$

### 11.4.2 Consequences of the power identities - capacitive and inductive modes

Based on the power-energy identities one may easily make several general observations regarding all types of evanescent waves. Since $P = 0$ energy is not transported along the guide. Energy is stored in the field and when averaged over a time period it is equally partitioned between the longitudinal and transverse fields. The reactive power $Q$ associated with a cutoff mode is proportional to the difference between the electric and magnetic pseudo-energies[7]. Owing to this one may easily classify an evanescent mode below cutoff as:

- the *capacitive* mode when

$$U_e > U_h, \quad Q < 0 \tag{11.4.15}$$

$$\mathcal{R}e\,\hat{U}_{ez} > \mathcal{R}e\,\hat{U}_{ht}, \quad \mathcal{R}e\,\hat{U}_{et} > \mathcal{R}e\,\hat{U}_{hz} \tag{11.4.16}$$

- or the *inductive* mode when

$$U_e < U_h, \quad Q > 0 \tag{11.4.17}$$

$$\mathcal{R}e\,\hat{U}_{ez} < \mathcal{R}e\,\hat{U}_{ht}, \quad \mathcal{R}e\,\hat{U}_{et} < \mathcal{R}e\,\hat{U}_{hz} \tag{11.4.18}$$

[7]Note that these properties are exactly the opposite of what we have established for propagating waves where the average electric and magnetic field energies were in balance.

From the above definitions it is immediately seen that because $\hat{U}_{ez} = 0$ for all $TE$ modes, these modes must be inductive below cutoff. Likewise $\hat{U}_{hz} = 0$ for $TM$ modes hence they are capacitive below cutoff.

With a proper normalization of field components $P = 0, Q = 1$ for inductive modes and $P = 0, Q = -1$ for capacitive modes. One may write this in explicit form as

$$\int_{\Omega} \hat{z} \cdot \left( \vec{E}_t \times \vec{H}_t^{\,*} \right) d\Omega = \pm j \tag{11.4.19}$$

The classification of modes into capacitive and inductive ones corresponds to the classification into forward and backward propagating waves in which the sign of the $zz$ component of Maxwell's stress tensor $\mathcal{R}e\,(\hat{U}_t - \hat{U}_z)$ provided the criterion for classification. It has to be noted that there is, however, a significant difference between the two classifications. In the case of propagating waves the sign of $\mathcal{R}e\,(\hat{U}_t - \hat{U}_z)$ was also related to the derivative of the propagation factor. A similar relation for $U_e - U_h$ and the frequency derivative of attenuation constant cannot be established without examination of suitable expression valid for cutoff waves. Unfortunately formula (11.2.7) cannot be used for this purpose as it was derived from variational expression (11.2.2) which is valid only for real $\beta$. In order to find out if the capacitive and inductive character of a wave also determines the sign of the derivative of the dispersion characteristics we need a new variational expression.

### 11.4.3 Variational expression for cutoff waves

Since we found out that cutoff waves have $P = 0$ and $4P = (\mathbf{M}x, x)$ it is evident now why formula (11.2.1) cannot be used to investigate cutoff waves. Instead, we have to apply a more general expression

$$\lambda = \frac{(\mathbf{A}x, \tilde{x})}{(\mathbf{M}x, \tilde{x})} \tag{11.4.20}$$

which was derived in Section 4.1.2 for the nonsymmetric problems. The application of functional (11.4.20) requires the knowledge of the solution $\tilde{x}$ of the transposed problem.

Let us see if we can find the $\tilde{x}$ directly from the solution $x$ of the original problem. To this end we have to consider the eigenvalue equations which are satisfied by cutoff modes. Since the structure is bidirectional we have two solutions

$$\mathbf{A}x = \beta \mathbf{M}x \tag{11.4.21}$$

$$\mathbf{A}x' = -\beta \mathbf{M}x' \tag{11.4.22}$$

However, as we already noted, because $\mathbf{A}$ and $\mathbf{M}$ are symmetric and $\beta$ is purely imaginary for cutoff modes the second equation is in fact the relation for the solution of the transposed problem

$$\mathbf{A}\tilde{x} = \beta^* \mathbf{M}\tilde{x} \tag{11.4.23}$$

We have $x' = \tilde{x}$ and we infer that the relation between the eigenfunction of the transposed problem $\tilde{x}$ and the eigenfunction of the original problem $x$ is identical to the relation between eigenfunctions of the complementary pair of modes $\pm\beta$. As shown in Sections 3.5 and 8.2, if $\mathbf{S}$ denotes the transformation satisfying

$$\mathbf{AS} = \pm\mathbf{SA} \qquad \text{and} \qquad \mathbf{MS} = \mp\mathbf{SM} \tag{11.4.24}$$

the eigenfunctions in the complementary pair are related via $x' = \mathbf{S}x$. Hence, knowing the eigenfunction of the original problem, we determine the eigenfunction of the transposed problem by simply putting $\tilde{x} = \mathbf{S}x$. Having found this relation we may write the variational expression for the propagation constant of modes below cutoff

$$\beta = \frac{(\mathbf{A}x, \mathbf{S}x)}{(\mathbf{M}x, \mathbf{S}x)} \tag{11.4.25}$$

The transformation relations for fields in the complementary pair in bidirectional guides have been derived in Section 8.2. We showed that these relations are known for a class of bidirectional structures possessing certain symmetries. Accordingly, (11.4.25) can be used for the investigation of such structures.

### 11.4.4 Cutoff modes in strictly bidirectional guides

To show how the variational expression can be used in the investigation of cutoff waves we shall henceforth concentrate on the most important category of strictly bidirectional guides (as defined at the beginning of Chapter 9). Other bidirectional structures may be analyzed in a similar manner. Before proceeding let us note that for strictly bidirectional guides

$$\mathcal{I}m\,\hat{U}_{et} = \mathcal{I}m\,\hat{U}_{ez} = \mathcal{I}m\,\hat{U}_{ht} = \mathcal{I}m\,\hat{U}_{hz} = 0 \tag{11.4.26}$$

Accordingly, the power-energy-momentum identities become

$$P = 0 \tag{11.4.27}$$

$$Q = \frac{\omega}{\alpha}(U_h - U_e) \tag{11.4.28}$$

$$U_{hz} - U_{ht} = U_{et} - U_{ez} \tag{11.4.29}$$

$$U_t = U_z \tag{11.4.30}$$

$$\mathcal{F} = \frac{\alpha}{\omega}(U_{et} - U_{ez} - U_{ht} + U_{hz}) \tag{11.4.31}$$

$$\mathcal{G} = 0 \tag{11.4.32}$$

Since also

$$U_e - U_h = 2(U_{ez} - U_{ht}) = 2(U_{et} - U_{hz}) \tag{11.4.33}$$

identity (11.4.28) is

$$Q = \frac{2\omega}{\alpha}(U_{ht} - U_{ez}) = \frac{2\omega}{\alpha}(U_{hz} - U_{et}) \tag{11.4.34}$$

We shall now see that some of the above identities follow directly from variational expression (11.4.25). For strictly bidirectional guides either a full six field component formulation or the specialized form

$$\mathbf{L_T}x = \beta\mathbf{M_T}x \tag{11.4.35}$$

where $\mathbf{L_T}$, and $\mathbf{M_T}$ are the reduced four field component operators, can be used.

$$\beta = \frac{(\mathbf{L_T}x, \mathbf{S}_{zT}x)}{(\mathbf{M_T}x, \mathbf{S}_{zT}x)} \tag{11.4.36}$$

where $\mathbf{S}_{zT}$ is given by

$$\mathbf{S}_{zT} = \begin{bmatrix} \underline{\underline{S}} & 0 \\ 0 & -\underline{\underline{S}} \end{bmatrix} \tag{11.4.37}$$

and

$$\underline{\underline{S}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{11.4.38}$$

Accordingly, the fields $\mathbf{S}_{zT}x$ are given by

$$\mathbf{S}_{zT}\{\vec{E}_t, \vec{H}_t\} = \{\vec{E}_t, -\vec{H}_t\} \tag{11.4.39}$$

Substituting the above relation into the formula (11.4.36) we get

$$\beta = \frac{\omega \int_\Omega \left(\epsilon_0 \vec{E}_t^{\,*} \cdot \underline{\underline{\epsilon}}_t \cdot \vec{E}_t - \mu_0 \vec{H}_t^{\,*} \cdot \underline{\underline{\mu}}_t \cdot \vec{H}_t - \mu_0\mu_{zz}|\vec{H}_z|^2 + \epsilon_0\epsilon_{zz}|\vec{E}_z|^2\right) d\Omega}{-2j\mathcal{I}m \int_\Omega \hat{z} \cdot \left(\vec{E}_t \times \vec{H}_t^{*}\right) d\Omega} \tag{11.4.40}$$

or using a shorthand notation

$$\beta = \omega\frac{U_{et} - U_{ht} - U_{hz} + U_{ez}}{-jQ} = \omega\frac{U_e - U_h}{-jQ} \tag{11.4.41}$$

Since for cutoff modes $\beta = -j\alpha$ then (11.4.41) is identical with (11.4.28).

In passing, we note that energy balance formula (11.4.30) follows directly from condition (3.2.15). Indeed, as we already noted, since the eigenvalue for cutoff waves is imaginary and the operators are symmetric, the inner product involving the relevant operators must vanish or

$$(\mathbf{L_T}x, x) = \omega \int_\Omega \left( \epsilon_0 \vec{E}_t^{\,*} \cdot \underline{\underline{\epsilon}}_t \cdot \vec{E}_t + \mu_0 \vec{H}_t^{\,*} \cdot \underline{\underline{\mu}}_t \cdot \vec{H}_t - \mu_0 \mu_{zz} |\vec{H}_z|^2 - \epsilon_0 \epsilon_{zz} |\vec{E}_z|^2 \right) d\Omega = 0 \tag{11.4.42}$$

Relation (11.4.42) can be written in a form of the energy balance formula for cutoff waves in strictly bidirectional guides

$$U_t = U_z \tag{11.4.43}$$

Having shown the application of the variational formula derived from the reduced formulation we now pass on to the investigation of the frequency behaviour of $\alpha$. In order to derive the expression for the derivative of the attenuation constant it is more convenient to use the six field component formula (11.4.25) with the reflection operator $\mathbf{S}_z$ defined in (8.2.14). The fields $\mathbf{S}_z x$ are given by

$$\mathbf{S}_z\{\vec{E}, \vec{H}\} = \{\vec{E}_t, -\vec{E}_z, -\vec{H}_t, \vec{H}_z\} \tag{11.4.44}$$

Using the variational property of (11.4.25) for fields satisfying relations (11.4.44) and strictly bidirectional guides, we get the following expression for the frequency derivative of the attenuation constant

$$\frac{\partial}{\partial\omega}\beta = \frac{\displaystyle\int_\Omega \left[ -\mu_0 \vec{H}^* \cdot \frac{\partial}{\partial\omega}\left(\omega \overleftrightarrow{\mu}\right) \vec{H} + \epsilon_0 \vec{E}^* \cdot \frac{\partial}{\partial\omega}\left(\omega \overleftrightarrow{\epsilon}\right) \vec{E} \right] d\Omega}{\displaystyle\int_\Omega \left( -\vec{E} \times \vec{H}^* + \vec{E}^* \times \vec{H} \right) \cdot \hat{z}\, d\Omega} \tag{11.4.45}$$

which may be written in the shorthand notation as

$$\frac{\partial}{\partial\omega}\alpha = \frac{W_{ht} - W_{hz} + W_{et} + W_{ez}}{Q} \tag{11.4.46}$$

with

$$W_{et} = \frac{\epsilon_0}{4} \int_\Omega \left[ \vec{E}_t^{\,*} \cdot \frac{\partial}{\partial\omega}\left(\omega \underline{\underline{\epsilon}}_t\right) \cdot \vec{E}_t \right] d\Omega \tag{11.4.47a}$$

$$W_{ez} = \frac{\epsilon_0}{4} \int_\Omega \left[ \frac{\partial}{\partial\omega}\left(\omega \epsilon_{zz}\right) |\vec{E}_z|^2 \right] d\Omega \tag{11.4.47b}$$

$$W_{ht} = \frac{\mu_0}{4} \int_\Omega \left[ \vec{H}_t^{\,*} \cdot \frac{\partial}{\partial\omega}\left(\omega \underline{\underline{\mu}}_t\right) \cdot \vec{H}_t \right] d\Omega \tag{11.4.47c}$$

$$W_{hz} = \frac{\mu_0}{4} \int_\Omega \left[ \frac{\partial}{\partial\omega}\left(\omega \mu_{zz}\right) |\vec{H}_z|^2 \right] d\Omega \tag{11.4.47d}$$

Relations (11.4.46), (11.4.41) can be combined yielding

$$\frac{\partial}{\partial \omega}\alpha = \frac{\alpha(W_{ht} - W_{hz} - W_{et} + W_{ez})}{\omega(U_h - U_e)} \tag{11.4.48}$$

Let us now compare the above expression with (11.1.10), which was derived for propagating waves. Unlike in the latter case the numerator in expression (11.4.48) can be zero so we can have inflection points on the dispersion characteristics below cutoff. On the other hand, for both expressions we cannot a priori exclude the zero values of the denominator and the derivatives of the dispersion characteristic can have singular points both below and above cutoff.

### Nondispersive media

For a nondispersive medium the pseudo-energies $U$ are equal to true energies $W$. Using this fact we may rewrite the relations obtained in the preceding section in the following form

$$\frac{\partial}{\partial \omega}\alpha = \frac{U_{ht} - U_{hz} - U_{et} + U_{ez}}{Q} \tag{11.4.49}$$

$$\frac{\partial}{\partial \omega}\alpha = \frac{\alpha(U_{ht} - U_{hz} - U_{et} + U_{ez})}{\omega(U_h - U_e)} \tag{11.4.50}$$

Using (11.4.33) the above expression can be transformed to

$$\frac{\partial}{\partial \omega}\alpha = \frac{\alpha(U_{ht} - U_{hz} - U_{et} + U_{ez})}{2\omega(U_{ht} - U_{ez})} = \frac{\alpha(U_{ht} - U_{hz} - U_{et} + U_{ez})}{2\omega(U_{hz} - U_{et})} \tag{11.4.51}$$

The energy relations derived so far for strictly bidirectional guides are identical with the expressions obtained by Chorney (1961) who used classical techniques. We shall now complement these with the formulae involving complex momentum. From (11.4.31) it is seen that the numerator in (11.4.49) is the negative of $\mathcal{F}$. As a result one gets

$$\frac{\partial}{\partial \omega}\alpha = -\frac{\omega \mathcal{F}}{\alpha Q} \tag{11.4.52}$$

or

$$\frac{\omega}{\alpha}\frac{\partial}{\partial \alpha}\omega = -\frac{Q}{\mathcal{F}} \tag{11.4.53}$$

In (11.4.53) we easily discover the analogue of the partial-power law (Brown's identity) expressed by (11.3.32). Because the mode is cutoff, $P = \mathcal{G} = 0$, but we see that taking the ratio of the imaginary parts of the Poynting vector and electromagnetic momentum we get the same form of the partial-power law. As a final comment let us note that we may now infer from (11.4.52) that the inflection points on the dispersion characteristic below cutoff are associated with the sign change of the imaginary part of the $z$ component of the momentum vector integrated over the guide cross-section. Similarly, the sign change of $Q$, which implies passing from the capacitive to inductive character of the mode (or vice versa), brings about the singular points of the derivative.

### 11.4.5 Critical points - strictly bidirectional guides

If the mode changes from evanescent to propagating, or vice versa, the propagation constant crosses the $\beta = 0$ plane. Accordingly, the properties of modes at the transition points are simply the combination of the relations already derived. In particular we may easily infer for a strictly bidirectional guide that the following relations must hold at the cutoff frequency.

$$U_e - U_h = U_z - U_t = P = Q = \mathcal{F} = \mathcal{G} = 0 \tag{11.4.54}$$

Moreover, because (11.3.13) and (11.4.33) hold, the $TE$ modes at cutoff must have

$$U_{et} = U_{hz} \tag{11.4.55}$$

Likewise, for the modes that are $TM$ at cutoff

$$U_{ht} = U_{ez} \tag{11.4.56}$$

Also, because $P = 0$ and $W_e + W_h \neq 0$, the derivative $\frac{\partial}{\partial \omega}\beta$ becomes unbounded which means that the group velocity is zero at the cutoff frequency.

At other critical points, with critical propagation factors, the modes are also purely $TE$ or $TM$ as shown in Section 9.4.3. It can easily be verified that the fields for $\omega = 0$ must satisfy relations (11.4.54), (11.4.55) and (11.4.56).

## 11.5 Complex waves

The reasoning presented in the previous section revealed that lossless nonbidirectional guides cannot support evanescent waves. Therefore only two types of waves, namely propagating and complex modes, are possible for these structures. As nonreciprocal materials are often used in practice, it is seen now that the complex waves are essential for the analysis of inhomogeneous waveguides and have unjustly been disregarded in practically all previous texts.

One consequence of the symmetry of operators $\mathbf{A}$ and $\mathbf{M}$ is that the complex waves in the lossless structures have to occur in complex conjugate pairs. In other words - if the structure is lossless then the existence of a complex mode with propagation constant $\beta$ implies the existence of the complementary mode with propagation constant $\beta^*$. This property of symmetric generalized eigenproblems was discussed in Section 3.5. Another general conclusion which can be drawn at this stage, based on the results of Section 3.5, is that since the 'energy norm' vanishes for complex eigensolutions of the generalized symmetric eigenproblems, the complex modes have to satisfy

$$(\mathbf{A}x, x) = (\mathbf{M}x, x) = 0 \tag{11.5.1}$$

or in the component form

$$\int_\Omega \left\{\mu_0 \vec{H}^*\left(\omega \overleftrightarrow{\mu}\right)\vec{H} + \epsilon_0 \vec{E}^*\left(\omega \overleftrightarrow{\epsilon}\right)\vec{E} - j\vec{H}^* \cdot \nabla_t \times \vec{E} + j\vec{E}^* \cdot \nabla_t \times \vec{H}\right\} d\Omega =$$

$$= 2\mathcal{R}e \int_\Omega \left(\vec{E} \times \vec{H}^*\right) \cdot \hat{z} d\Omega = 0 \tag{11.5.2}$$

The second of the above forms obviously expresses the power transported along the guide and can be written as

$$P = 0 \tag{11.5.3}$$

This situation is reminiscent of that which we encountered for the case of cutoff waves. Complex modes, therefore, do not take part in the energy transport along the guide. To explain their role in guided wave propagation it is necessary to investigate the properties which follow from the conservative properties (7.4.16), (7.5.10). Unlike in the previous cases we cannot simplify these expressions by assuming that either part of the propagation factor is real. We shall, however, restrict the analysis to the case of steady harmonic state $\mathcal{I}m\,\omega = 0$. Accordingly, the starting point for the investigation of complex waves uses the following identities

$$(\mathcal{I}m\,\beta - j\mathcal{R}e\,\beta)(P + jQ) = -2j\omega(\hat{U}_{et} - \hat{U}_{hz}) \tag{11.5.4}$$

$$(\mathcal{I}m\,\beta + j\mathcal{R}e\,\beta)(P + jQ) = -2j\omega(\hat{U}_{ez} - \hat{U}_{ht}) \tag{11.5.5}$$

and

$$\omega(\mathcal{G} - j\mathcal{F}) = 2(\beta\hat{U}_{et} + \beta^*\hat{U}_{ez}) \tag{11.5.6}$$

$$\omega(\mathcal{G} - j\mathcal{F}) = 2(\beta^*\hat{U}_{ht} + \beta\hat{U}_{hz}) \tag{11.5.7}$$

From (11.5.4) and (11.5.5) the following identities follow

$$P = 0 \tag{11.5.8}$$

$$\mathcal{R}e\,(\hat{U}_t - \hat{U}_z) = 0 \tag{11.5.9}$$

$$Q = \frac{\omega}{\mathcal{R}e\,\beta}\mathcal{I}m\,(\hat{U}_t - \hat{U}_z) \tag{11.5.10}$$

$$Q = \frac{\omega}{\mathcal{I}m\,\beta}(U_h - U_e) \tag{11.5.11}$$

Combining the last two identities gives

$$\mathcal{R}e\,\beta(U_h - U_e) = \mathcal{I}m\,(\beta)\mathcal{I}m\,(\hat{U}_t - \hat{U}_z) \tag{11.5.12}$$

The momentum identities yield

$$\omega\mathcal{G} = \beta(\hat{U}_{et} + \hat{U}^*_{ht}) + \beta^*(\hat{U}_{ez}) + \hat{U}^*_{hz})) \tag{11.5.13}$$

$$j\omega\mathcal{F} = \beta(\hat{U}^*_{ht} - \hat{U}_{et}) - \beta^*(\hat{U}_{ez}) - \hat{U}^*_{hz})) \tag{11.5.14}$$

### 11.5.1 Strictly bidirectional guides

A considerable simplification of the above identities occurs when the medium is strictly bidirectional. In that case

$$\hat{U}^*_{ht} = \hat{U}_{ht} = U_{ht} \tag{11.5.15}$$

$$\hat{U}^*_{hz} = \hat{U}_{hz} = U_{hz} \tag{11.5.16}$$

also the transverse and longitudinal parts of electric pseudo-energy densities are real. Accordingly, the identities for the complex waves in strictly bidirectional waves become

$$Q = 0 \tag{11.5.17}$$

$$U_e = U_h \tag{11.5.18}$$

$$U_t = U_z \tag{11.5.19}$$

As a corollary we also find

$$U_{ht} - U_{et} = U_{ez} - U_{hz} \tag{11.5.20}$$

The above identities, substituted into (11.5.13) and (11.5.14), under the condition $\beta = \mathcal{R}e\,\beta - j\mathcal{I}m\,\beta$, yield

$$\frac{\mathcal{R}e\,\beta}{\omega} = \frac{\mathcal{G}}{2U_t} = \frac{\mathcal{G}}{2U_z} \tag{11.5.21}$$

$$\frac{\mathcal{I}m\,\beta}{\omega} = \frac{\mathcal{F}}{2(U_{ez} - U_{hz})} = \frac{\mathcal{F}}{2(U_{ht} - U_{et})} \tag{11.5.22}$$

From the above derivations it is clearly seen that a complex wave in strictly bidirectional guide appears to behave, simultaneously, as a propagating wave ($U_e = U_h$) and as a cutoff wave ($U_z = U_t$). However, since $P = Q = 0$ the wave carries neither real nor reactive power. This raises the question what is the role of complex waves in the guided wave propagation and analysis. To explain this one has to recall that the complex eigensolutions are not orthogonal. Consequently, even though[8]

$$P_{ii} = Q_{ii} = 0 \tag{11.5.23}$$

in general

$$P_{ij} \neq 0 \quad \text{and} \quad Q_{ij} \neq 0 \tag{11.5.24}$$

for a pair of complex modes corresponding to complex conjugate eigenvalues. In other words, the electric field of one wave can couple to the magnetic field of the other wave. To examine the consequences of this fact let us investigate pairs of complex waves in a strictly bidirectional guide loaded with a *reciprocal* medium. Complex waves in such guides can occur in foursomes $\beta, \beta^*, -\beta, -\beta^*$. Let us examine the interaction of a pair $\beta_1 = \beta$ and $\beta_2 = -\beta^*$. According to the results obtained in Section 8.2.2, the fields of the two waves are related

$$\vec{E}_{t2} = \vec{E}_{t1}^* \quad \text{and} \qquad \vec{H}_{t2} = -\vec{H}_{t1}^* \tag{11.5.25}$$

Accordingly

$$P_{12} + jQ_{12} = \frac{1}{2}\int_\Omega \left(\vec{E}_{t1} \times \vec{H}_{t2}^*\right) \cdot \hat{z}\, d\Omega = -\frac{1}{2}\int_\Omega \left(\vec{E}_{t1} \times \vec{H}_{t1}\right) \cdot \hat{z}\, d\Omega \tag{11.5.26}$$

$$P_{21} + jQ_{21} = \frac{1}{2}\int_\Omega \left(\vec{E}_{t2} \times \vec{H}_{t1}^*\right) \cdot \hat{z}\, d\Omega = \frac{1}{2}\int_\Omega \left(\vec{E}_{t1}^* \times \vec{H}_{t1}^*\right) \cdot \hat{z}\, d\Omega \tag{11.5.27}$$

It is evident that

$$P_{21} + jQ_{21} = -(P_{12} + jQ_{12})^* \tag{11.5.28}$$

As a result, since

$$(P_{12} + jQ_{12}) + (P_{21} + jQ_{21}) = 2jQ_{12} \tag{11.5.29}$$

such a pair behaves reactively as a whole and is capable of storing energy. It must therefore contribute to the overall energy balance. This result, obtained also by Chorney (1961), is a generalization of the conclusion reached for complex waves in guides containing isotropic material (Omar and Schünemann; 1987; Jabłoński; 1994).

## 11.6 Dispersion characteristics - strictly bidirectional guides

We have now established enough facts to be able to link the dispersion characteristics to the energy relations. Introducing the mode nomenclature in Section 2.2.1

[8] We restore the indices $ij$ to distinguish different modes.

we divided all waves into normal and anomalous waves. The criterion used is the behaviour of the propagation factor versus frequency. Let us see how these categories of waves are related to the physical quantities discussed in this chapter. We shall restrict our investigation to strictly bidirectional guides loaded with a dispersion free material. If the wave is propagating then the normal wave is the wave for which the phase factor increases with frequency. Accordingly, it is a forward wave. An anomalous wave shows the opposite behaviour and is therefore the backward wave. These two categories of waves are associated with different directions of the Poynting vector and the sign of the $zz$ component of the Maxwell stress tensor changes. It is interesting to observe, however, that the sign of the electromagnetic momentum is always positive for both categories of waves. These observations are summarized below:

- Forward wave

$$U_t > U_z, \quad U_{et} > U_{hz} \quad U_{ht} > U_{ez}, \quad \mathcal{G} > 0 \tag{11.6.1}$$

- Backward wave

$$U_t < U_z, \quad U_{et} < U_{hz} \quad U_{ht} < U_{ez} \quad \mathcal{G} > 0 \tag{11.6.2}$$

The behaviour of waves below cutoff is different. The waves are either capacitive or inductive. Normal waves show the decreasing attenuation factor for increasing $\omega$. The converse is true for an anomalous wave. For cutoff waves one cannot infer the capacitive or dispersive character of a mode based solely on the investigation of the dispersion characteristic. However, an observation can be made regarding the transition from normal to anomalous wave. This transition is associated with the change of sign of the reactive parts of either the momentum $\mathcal{F}$ or the Poynting vector $Q$. The former change brings about the inflection point on the dispersion characteristic, while the latter is associated with either cutoff or the change of the mode character from inductive to capacitive or vice versa.

As for the complex waves the most important properties are expressed by

$$U_e = U_h, \; Q = 0, \; U_t = U_z, \; P = 0 \quad \text{but} \quad \mathcal{G} \neq 0, \mathcal{F} \neq 0 \tag{11.6.3}$$

The properties of complex waves should be now compared with the expression valid for critical points

$$U_e = U_h, \; Q = 0, \; U_t = U_z, \; P = 0 \quad \text{and} \quad \mathcal{G} = 0, \mathcal{F} = 0 \tag{11.6.4}$$

It is seen that except for the nonzero momentum, a complex wave has properties similar to those which a mode acquires when passing through cutoff.

These observations lead to the following conclusions regarding characteristic points on the dispersion curves.

- The derivative of the dispersion curve may have singularities occurring when either $P$ or $Q$ becomes zero. The vanishing of $P$ takes place when $U_e = U_h$. Similarly, vanishing of $Q$ occurs for $U_t = U_z$. Unless $\beta = 0$, singularities in the derivative of a dispersion characteristics bring about the bifurcation of a curve (Silbergleit and Kopilevich; 1983, 1981). Accordingly, if $P$ or $Q$ changes sign when passing through such a singularity, one gets a bifurcation point. Since the wave at such a point has the power-energy properties of complex waves, bifurcations mark the onset of a complex wave. Note that $P$ passes through zero when the mode changes the character from inductive to capacitive or vice versa. Similarly, $Q$ has to go through zero when one passes from the forward wave into a backward one.

- $P$ and $Q$ are simultaneously zero at cutoff. This implies that the derivative is infinite at characteristic frequencies.

- The slope of the dispersion curve is bounded from below for the propagating waves. Inflection points on the dispersion curves are possible only below cutoff. Such inflection points are associated with the sign change of the reactive part of electromagnetic momentum.

As an illustration of the foregoing discussion let us briefly discuss the characteristic points at a typical dispersion graph shown in figure 11.1. The structure actually used in the computation was a circular cylinder loaded with a high permittivity dielectric rod, but the behaviour of the curves is characteristic of any strictly bidirectional guide inhomogenously loaded with lossless and dispersionless material.

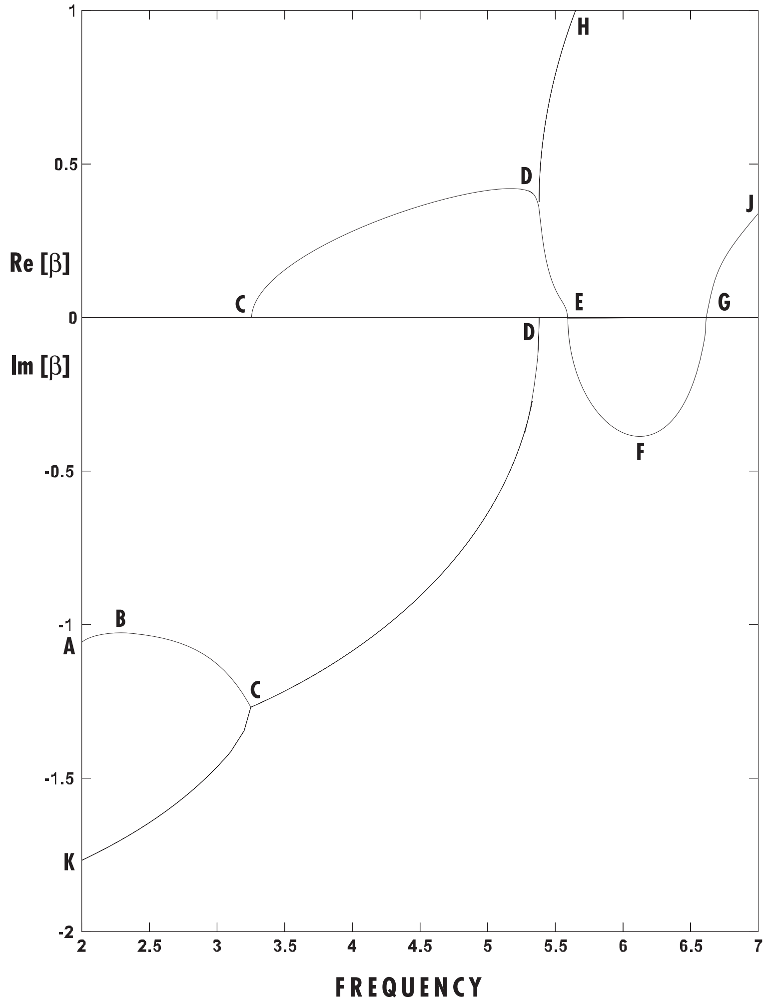
- Between points, A-B, K-C, F-J and D-H we observe normal waves. Anomalous waves exist between points B-C and D-F.

- There are two inflection points (B,F) - both below cutoff. At these points the reactive part of momentum $\mathcal{F}$ changes sign.

- There are two singular points C and D. At each of these points the derivative is infinite and bifurcation occurs. In each case the singularity is associated with a degeneracy of normal and anomalous wave. Judging from the derivative being singular we may infer that $Q$ and $P$ must be zero, respectively, at points C and D.

- A complex wave exists from point C to D. For the complex wave, $U_t = U_z$ and $U_e = U_h$. As one passes through point D, $(U_t - U_z)$ splits into two branches. $U_t - U_z$ is negative for branch D-E (an anomalous or backward

*Figure 11.1: Dispersion diagram for an inhomogeneously filled waveguide plotted using the convention in which the imaginary and real parts of $\beta$ are shown in the lower and upper part of the graph, respectively. The letters mark the characteristic points which can be explained by the energy relations.*




wave) and positive for D-H (a normal or forward wave). Traversing point C gives the splitting into branches with positive or negative $U_e - U_h$ i.e. into capacitive and inductive wave. One of these waves is normal and the other is anomalous. It is clearly seen that an infinite derivative gives a bifurcation point and marks the onset of a complex wave.

- Points E and G are cutoffs. Here $U_t - U_z$ and $U_e - U_h$ become simultaneously zero and the modes are purely $TE$ or $TM$.

- Between points E and G, a mode changes from anomalous wave to the normal wave. However, since the derivative is finite within the open interval (E,G) the mode does not change the character from capacitive to inductive or vice versa.

## 11.7 Concluding remarks - complex modes at fixed $\beta$

The results regarding dispersion characteristics presented in this chapter concern exclusively the case of steady harmonic state and $\beta = g(\omega)$ curves. As noted in Section 2.2.1 one may also investigate inverse curves by exchanging the ordinate and abscissa. The dispersion characteristics emerge then in the form of $\omega = f(\beta)$ curves. To obtain these curves one may start from the $\omega$ formulation. However, the solution of such problems will be guaranteed to yield only real values of $\omega$ for $\mathcal{I}m\,\beta = 0$. This is the consequence of one of the operators in a generalized eigenvalue problem being symmetric and definite only for real $\beta$ (see Section 7.3.3). If one is interested in expressing the field for fixed $\mathcal{I}m\,\beta \neq 0$ in terms of any possible oscillations then it is necessary to also admit nonharmonic solutions corresponding to complex $\omega$. These solutions may be regarded as a different types of complex modes. We shall not investigate this category of complex oscillations in detail but only briefly outline the methodology and give some of the conclusions for strictly bidirectional waveguides. The starting point is at the pencil representation. Taking the plain or weighted inner product of pencil (9.2.1) or (9.2.3) with $\underline{\Psi} = [\vec{E}_t, \vec{H}_t]^T$, under the assumption that both $\beta$ and $\omega$ can be complex, and using expressions (9.4.7) or (9.4.6), one gets the following system of equations

$$\omega U_t - \beta P - \omega^* U_z = 0 \tag{11.7.1}$$

$$\beta U_t - \omega \mathcal{G} + \beta^* U_z = 0 \tag{11.7.2}$$

These expressions may now be considered simultaneously as a homogeneous system of equations. Such a system has been investigated by Gureev (1990), who restricted his analysis to guides containing only isotropic media and reached the following conclusions:

1. No waves can simultaneously have $\mathcal{R}e\,\beta \neq 0$ and $\mathcal{R}e\,\omega = 0$ or $\mathcal{I}m\,\omega \neq 0$ and $\mathcal{I}m\,\beta = 0$.

2. At $\omega = 0$ all waves are evanescent ($\mathcal{R}e\,\beta = 0$).

3. Critical frequencies, corresponding to the condition $\beta = 0$ are real.

4. For waves with $\mathcal{R}e\,\omega \neq 0, \mathcal{R}e\,\beta = 0$ or $\mathcal{I}m\,\omega = 0, \mathcal{I}m\,\beta \neq 0$, the following relation holds
$$U_t - U_z = 0 \tag{11.7.3}$$

5. Waves $\mathcal{I}m\,\omega \neq 0, \mathcal{I}m\,\beta \neq 0$ do not transport energy $P = 0$.

6. The time-averaged $z$ component of the linear momentum density vanishes ($\mathcal{G}$ =0) for $\mathcal{R}e\,\omega \neq 0, \mathcal{R}e\,\beta = 0$.

7. The time-averaged energies are related by the equation
$$P\mathcal{G} = U_t^2 - U_z^2 \tag{11.7.4}$$

8. Any three time-averaged energy characteristics of a wave with $\mathcal{I}m\,(\omega\beta)$ can be expressed by the fourth, for instance
$$U_z = \frac{\mathcal{I}m\,(\omega^*\beta)}{\mathcal{I}m\,(\omega\beta)}U_t, \quad P = 2\frac{\mathcal{R}e\,\omega\mathcal{I}m\,\omega}{\mathcal{I}m\,(\omega\beta)}U_t, \quad \mathcal{G} = 2\frac{\mathcal{R}e\,\beta\mathcal{I}m\,\beta}{\mathcal{I}m\,(\omega\beta)}U_t \tag{11.7.5}$$

The above conclusions, although derived for isotropic media, include many of the results which have been obtained earlier for anisotropic case. An important point to note, however, is that now also nonharmonic states are allowed.

Moreover, the extension of the pencils to the case of strictly bidirectional guides introduces changes only into expressions defining $U_t, U_z, \mathcal{G}$; equations (11.7.1) and (11.7.2) retain, however, their form. Consequently Gureev's conclusions can be directly carried over to a larger class of guides. As a final observation we note that one may also form another set of equations by taking the inner product with $\underline{\Phi} = [\vec{E}_t, -\vec{H}_t]^T$, rather than $\underline{\Psi} = [\vec{E}_t, \vec{H}_t]^T$. These equations result in expressions involving $Q$ and $\mathcal{F}$ and can be used to determine the conditions under which complex values of $\omega$ are possible.

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