Paraelasticity

A. Niemunis · L. F. Prada-Sarmiento · C. E. Grandas-Tavera

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Abstract A hysteretic, fully reversible model with rate independent damping and with variable secant stiffness is proposed for the 3D case. A reversible dilatancy-contractancy effect can be optionally used as described in the companion paper [11]. The present paper presents the basic aspects of the model only. The definition of a strain-path reversal is given and the handling of the past reversals using the stack structure is proposed.

Keywords Paraelasticity · Hysteresis · Stack of reversals

1 Introduction

Purely hysteretic constitutive behavior excludes cumulative effects but allows for a non-unique stress strain dependence. We distinguish between a strain loop and a strain cycle, Fig. 1. Both denote closed strain paths i.e. the initial strain and the final strain are identical. A cycle is a special kind of loop which is commenced at a reversal, i.e. at a turning point of the strain path.

In paraelasticity (PE), strain cycles must lead to closed stress paths but strain loops need not.

A non-unique relation between stress and strain is necessary to capture the damping (energy dissipation). The paraelastic model introduces the concept of a strain span $e$ which is the sum of strain increments measured from the most recent reversal. This concept was originally proposed in [6, 12]. PE is presented as a stand-alone model here, but it can improve the small-strain stiffness behavior of other models, in our case the hypoplasticity (HP) [19] or viscohypoplasticity [9, 10].

Compared to [6] the present version of paraelasticity contains two novel elements, namely:

- the definition of the distance from the reversal point and the update rules have been modified
- the reversible dilatancy - contractancy behavior has been added [11]

In this first article the basic concept of strain path reversals and their update rules are defined and presented in Section 2. The performance of paraelasticity in the description of shear degradation and in the simulation of damping ratio curves is discussed in Section 3.3. Numerical examples for the 1D model and calibration of material constants are given in Sections 3.3.2-3.3.3.
The stack of reversals and its update rules

The paraelastic description of the hysteretic strain-stress behavior is based on the strain reversals. They are points in the strain space which may be generated by a sudden turn in the strain increments and which may be deleted by overloading\(^2\). In this section we describe the reversals and their properties:

- condition for a generation and deletion of a strain path reversal
- loading direction, strain span, distance between reversals
- loading circle, reversal circle
- stack of reversals and push, pop and drag operations defined on the stack
- root reversal
- splitting of the strain span

The sequence of reversals describes the history of deformation. The PE stress-strain relation is path dependent because it depends on the stack.

2.1 Stack of reversals

Like most nonlinear constitutive models the paraelastic model works with strain increments \( \Delta \epsilon \). Starting from the most recent reversal \( R_1 \) (defined further) the strain increments are added in a so-called strain span \( \epsilon = \epsilon - \epsilon^{R1} \). The model describes stress span \( \sigma = \sigma - \sigma^{R1} \) as a function of the strain span. A reversal is established when the distance\(^3\) (denoted as \( d_{ARR1} \)) between the current state \( A \) and the most recent reversal \( R_1 \) starts to decrease. The oldest reversal, a so-called root reversal \( L \) must be specially initialized. The youngest reversal \( R_1 = \{ \epsilon^{R1}, \sigma^{R1}, N^{R1}, \epsilon^{R1}, d_{RR1}, P_{RR1} \} \) consists of the following information: strain \( \epsilon^{R1} \), stress \( \sigma^{R1} \), loading direction \( N^{R1} \), void ratio \( \epsilon^{R1} \) corresponding to this reversal, the corrected pressure \( P_{RR1} \) accounting for dilatancy/contractancy effects, see \([11]\), and the distance \( d_{RR1} \) from \( R_1 \) to the one but last reversal \( R_2 = \{ \epsilon^{R2}, \sigma^{R2}, N^{R2}, \epsilon^{RR2}, d_{RR2}, P_{RR2} \} \). Older reversals \( R_3 = \{ \epsilon^{R3}, \sigma^{R3}, N^{R3}, \epsilon^{RR3}, d_{RR3}, P_{RR3} \} \) etc. have the same structure. The root reversal \( L = \{ \epsilon^L, \sigma^L, N^L, \epsilon^L, d_{LR}, P_{RLR} = 0 \} \) is indelible and needs a special treatment. Its distance \( d_{LR} \) is a material constant defining the size of the paraelastic region. The current state is stored in an analogous collective variable \( A = \{ \epsilon, \sigma, N, e^A, d_{ARR1}, P_{ARR1} \} \). Sometimes it is necessary to distinguish between the current states \( A^n = \{ \epsilon^n, \sigma^n, N^n, \epsilon^n, d_{ARR1}, P_{ARR1} \} \) and \( A^{n+1} = \{ \epsilon^{n+1}, \sigma^{n+1}, N^{n+1}, \epsilon^{n+1}, d_{ARR1}, P_{ARR1} \} \) before and after the current strain increment, respectively.

The current state and all reversals are kept in a LIFO\(^5\) list \( \{ A, R_1, R_2, \ldots, L \} \) of length \( l \) termed the stack of reversals.

A turn in the strain path may generate a new reversal (a reversal is pushed onto the stack) and during a monotonc (over-)loading the most recent reversals are removed from the stack (popped).

In a series of cycles with decreasing amplitude (a shakedown) all reversals are meaningful and the stack may become too large for the computer memory. In order to restrict the memory requirement of the model a consolidation of stack may be performed.

2.2 Loading and reversal circles

The loading circle (a hyper-sphere in the 6D strain space) is uniquely defined by the following conditions, Fig. 2:

- it passes through \( \epsilon^{R1} \) and \( \epsilon^{n+1} \)
- it has a unit outer normal \( N^{R1} \) at \( \epsilon^{R1} \)

\[ \text{Fig. 2 Loading circle (small) and reversal circle (large) in the strain space generated by the strain path with reversals.} \]

Given a reversal \( R_1 = \{ \epsilon^{R1}, \sigma^{R1}, N^{R1}, \epsilon^{R1}, d_{RR1}, P_{RR1} \} \) and the updated current strain \( \epsilon^{n+1} = \epsilon^n + \Delta \epsilon \) we calculate the center \( \epsilon \) and the radius \( r \) of the loading circle from the equation system

\[
\begin{aligned}
\epsilon &= \epsilon^{R1} - r N^{R1} \\
|r| &= ||\epsilon^{n+1} - \epsilon|| \tag{1}
\end{aligned}
\]

The distance between \( A^{n+1} \) and \( R_1 \) is defined as the diameter \( d_{ARR1} = 2r \) of the loading circle, i.e. it can be calculated from

\[
d_{ARR1} = -\frac{\epsilon : \epsilon}{\epsilon : N^{R1}} \quad \text{with} \quad \epsilon = \epsilon^{n+1} - \epsilon^{R1} \tag{2}
\]

\(^2\) The concept of overloading is explained in Section 2.3.

\(^3\) Definition is given further in this section.

\(^4\) At this moment the notation with long indices may seem clumsy, however, it will become natural in more complex cases.

\(^5\) Last In, First Out.
rather than from the Euclidean norm $\|\epsilon^{n+1} - \epsilon^R1\|$. Equation (2) is valid for $\epsilon^N1 < 0$ only and $d_{AR1} = \infty$ must be used\(^6\) otherwise. All $d_{AR1}$ must be smaller than the diameter of the root circle $d_L$ (a material constant). The loading direction $N^{n+1}$ is defined as a unit normal with respect to the loading circle

$$N^{n+1} = [\epsilon^{n+1} - \epsilon]^{-1} = [\epsilon^{n+1} - (\epsilon^R1 - rN^R1)]^{-1} \quad (3)$$

at $A^{n+1}$.

Unloading occurs if $d_{AR1}^{n+1} < d_{AR1}^n$. In such case the loading circle passing through $A^n$ becomes a reversal circle with the properties $\{\epsilon^n, \sigma^n, N^n, e^n_A, d^n_{AR1}, P^n_{AR1}\}$ memorized as the youngest reversal $R_1$. Moreover, the indices of the older reversals in the stack are increased by one. We call it the push operation.

The distance between any two consecutive reversals $R_i+1, R_i$ is denoted as $d_{RR1}$. It can also be calculated from (2) using $\epsilon = \epsilon^{R_i+1} - \epsilon^R1$ and $N^R1$. Note that $d_{AR1}$ satisfies the axioms of distance:

- $d_{AR1} \geq 0$ (non-negativity)
- $d_{AR1} = 0$ iff $A = R_1$ (identity of indiscernibles)
- symmetry $d_{AR1} = d_{RA1}$
- $d_{AR1} + d_{AR2} \geq d_{RR1}$ (triangle inequality)

### 2.3 Update rules for the stack

The calculation of the paraelastic model is performed applying small\(^7\) strain increments $\Delta \epsilon$. Given $\Delta \epsilon$ we may find the new state $A^{n+1}$ using $\epsilon^{n+1} = \epsilon^n + \Delta \epsilon$, $e^{n+1} = e^n + \text{tr} (\Delta \epsilon)/(1 + e^n)$, Equations (2) and (3) for $d_{AR1}^{n+1}$ and $N^{n+1}$ respectively. The stress $\sigma^{n+1}$ and dilatancy/contractancy pressure $P_{AR1}^{n+1}$ are calculated as presented in the next subsections.

**Under loading conditions**

$$d_{AR1}^{n+1} > d_{AR1}^n \quad \text{and} \quad d_{AR1}^{n+1} < d_{RR1} \quad (4)$$

the diameter of the loading circle is monotonically increasing. No reversals are either generated or deleted.

**Under unloading condition**

$$d_{AR1}^{n+1} < d_{AR1}^n \quad (5)$$

the current loading circle (passing through $R_1$ and $A^n$) is memorized as the most recent reversal circle and a new loading circle develops from $A^n$. Moreover, the indices $R_i$ of the remaining reversals are incremented by one. We say that $A^n$ is pushed onto the stack, Fig. 3. The new (usually very small) loading circle is passing through $A^n$ and $A^{n+1}$ and has a common outer normal direction with the new reversal circle at $\epsilon^R1$.

Under overloading condition

$$d_{AR1}^{n+1} > d_{RR1} \quad (6)$$

the current loading circle outgrows the most recent reversal circle defined between $R_1$ and $R_2$ with diameter $d_{RR} = d_{RR1}$

The reversal point $R_1$ is swept out of the material memory and the indices of the remaining reversals are decreased by one, Fig. 3. This update of stack is termed pop. If the distance from $A^{n+1}$ to the new $R_1$ is larger than the new $d_{RR1}$, the new $R_1$ must be popped too. The reversals are popped until the loading condition $d_{AR1}^{n+1} < d_{RR1}$ is reached or until the root reversal $L$ is overloaded (it cannot be popped). Note that unloading cannot occur during multiple pop (when several reversals are popped by a single increment $\Delta \epsilon$). It can be shown that $d_{AR1}$ remains a continuous function of strain upon a single or multiple pop. In other words, an infinitesimally small increment $\Delta \epsilon$ may pop a reversal but $d_{AR1}$ with respect to new $R_1$ remains almost identical as $d_{AR1}$ with respect to the old one.

### Figure 3: The strain path creating new reversal circles upon unloading (3UL) or swallowing them upon loading (3L).

The diameter $d_L$ of the outermost circle corresponds to the root reversal $L$ which cannot be popped. Therefore, if the current distance $d_{AR1}$ exceeds $d_L$, a special dragging procedure is activated. Given a strain increment $\theta \Delta \epsilon$ that surpassed the root circle $d_{AR1}^{n+1} > d_L$ we distinguish a part $\theta \Delta \epsilon$ of this increment that lies within the root circle and another part $(1 - \theta) \Delta \epsilon$ that protrudes outside of it (Fig. 4). The decomposition of the strain increment is termed split.

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\(^6\) $d_{RR1}^{n+1} = 10d_L$ is sufficient for practical purposes.

\(^7\) Large increments may lead to inconsistencies in the constitutive description. One can show that within a single large increment a new unloading surface may appear and then be erased. The constitutive consequences of this are visible only if this increment is calculated with several small sub-increments.
The conditions for push, pop and drag operations can be summarized as follows

\[
\begin{align*}
&\begin{cases}
   d_{AR1}^{n+1} < d_{AR1}^n : & \text{renumber } R_{i+1}^{n+1} := R_i^n, \\
   & \text{and add } R_{i+1}^{n+1} := A^n, \\
   l^n > 2 \text{ and } d_{AR1}^{n+1} > d_{RR1}^n : & \text{erase } R_1^n, \\
   & p^{n+1} = l^n - 1, \\
   L = R_1 \text{ and } d_{AR1}^{n+1} > d_L : & \text{dragging}
\end{cases}
\end{align*}
\]

(7)

The index \(i = 1, \cdots, L\) denotes the sequence of reversals stored in the stack. Conditions for push and pop operations are shown schematically in Fig. 5.

If \(A^n\) lies inside the root circle and \(A^{n+1}\) lies outside of it, the increment \(\Delta \varepsilon\) must be split into two portions: the inner paraelastic portion \(\Delta \varepsilon^{PE}\) and the protruding dragging portion \(\Delta \varepsilon^{HP}\).

\[
\Delta \varepsilon = \Delta \varepsilon^{PE} + \Delta \varepsilon^{HP} = \theta \Delta \varepsilon + (1 - \theta) \Delta \varepsilon
\]

(8)

We may easily determine \(\theta\) as shown in Appendix A. Under dragging condition,

\[
d_{AR1}^{n+1} > d_L
\]

(9)

\(L\) is proposed to move parallel to \(A\) with identical increments of stress \(\Delta \sigma = \Delta \sigma^L\) and strain \(\Delta \varepsilon = \Delta \varepsilon^L\). The loading direction \(N\) remains unchanged while dragging.

An additional constitutive rule is required for the first unloading after dragging, i.e., for

\[
d_{AR1}^n = d_L \text{ and } d_{AR1}^{n+1} < d_L
\]

(10)

A usual push operation would add a new state \(A^{n+1}\) on the top of the stack keeping \(A^n\) as a reversal \(R_1\) with \(d_{AR1}\), and preserving \(L\) with the same size \(d_L\). However, only one reversal point with \(d_L\) is allowed for. Therefore \(L\) is removed from the stack and \(R_1\) is treated as the new \(L\). This special case is handled by the push algorithm. A MATHEMATICA script that calculates the push, pop and split procedures for the isomorphic P-Q space is given in Appendix B.

2.4 Initialization of the stack

The initialization of the stack should represent the loading history of the soil. The paraelastic stack must consist at least of two components \(\{A, R_1 = L\}\) which define the root reversal circle and the current loading circle.

If the soil is freshly pluviated, as it would be the case of a sample prepared in the laboratory, the normal vectors defining the paraelastic circles ought to be aligned along the vertical direction, as shown in Fig. 6, left.

Soils subjected to small cyclic perturbations should be initialized with a series of cycles with smaller amplitudes (shakedown) about the \(K_0\) direction. This shakedown condition is illustrated in Fig. 6, right.

Examples of the aforementioned initializations of the stack are written in the form of user’s routines\(^8\) for ABAQUS\(^9\).

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8 These Fortran95 routines can be obtained directly from the authors.
9 ABAQUS is a reg. trademark of Simulia Inc.
The initial stress $\sigma^A$, initial void ratio $e$, and the initial strain $\epsilon = 0$ are usually prescribed. The reversals for the shake down process up to the root reversal are unknown and would have to be calculated backwards. Instead one may perform a forwards calculation starting from $L = A$ and then correct the starting position $L$ until the desired state $A$ is reached.\(^{10}\)

### 3 Stress-strain relation and PE stiffness

In this section we interrelate the strain span $e = \epsilon - \epsilon^{R1}$ and the stress span $s = \sigma - \sigma^{R1}$. For any monotonously growing strain path with $d^{AR1}_{\text{span}} > d^{AR1}_{\text{span}}$, the stress-strain relationship is proposed in the following form

$$\mathbf{s} = H : \mathbf{e} = (1 - f d^{X}_{\text{AR1}}) \mathbf{E} : \mathbf{e},$$

wherein $d^{AR1}_{\text{span}}$ is given in (2) and $\mathbf{E}$ is the isotropic elastic stiffness tensor

$$\mathbf{E} = \lambda \mathbf{I} + 2\mu \mathbf{I}$$

with the Lamé constants\(^{11}\) $\lambda$ and $\mu$. For applications restricted to small $d^{AR1}_{\text{span}}$, the relation (11) is unique and can be inverted. The tangential stiffness (Jacobian) obtained from (11) is

$$s' = \frac{\partial \mathbf{s}}{\partial \mathbf{e}} = H - f \chi d^{X}_{\text{AR1}} \mathbf{E} : \mathbf{e} \eta \quad \text{with} \quad \eta = \frac{\partial d^{AR1}}{\partial \mathbf{e}}$$

The material constants $f > 0$ and $\chi > 0$ must guarantee that both the secant stiffness $(1 - f d^{X}_{\text{AR1}}) \mathbf{E}$ in (11) and the tangential stiffness (13) are positive. Tensor $\eta$ is outer normal to the loading circle at $A$ as discussed further, see Fig. 17. For a rough estimation of the restrictions on $f$ and $\chi$ we may use the inequality (52) or (59) derived in Appendix C.

#### 3.1 Negative dissipation

The elastic part $\mathbf{E}$ is barotropic (pressure dependant) and it has not been derived from any elastic potential. Therefore one could expect that the model may violate the Second Law of thermodynamics. However, $\mathbf{E}$ is based on the pressure in the middle of the root circle and hence it remains constant unless dragging occurs. In other words, in pure PE range the pressure dependence of $\mathbf{E}$ cannot lead to a perpetual motion of the second kind.

Independently, a non-unique $\sigma(\epsilon)$ relation could also lead to the violation of the Second Law. Numerically we attempted to construct a cycle (closed strain and stress) corresponding to minimum dissipation (possibly negative). In spite of varying the form of the cycles, the starting position, the sense of circulation and the number of reversals we could never obtain negative dissipation of energy. None of the tested cycles could produce mechanical work from heat. A formal mathematical proof of the compliance of PE with the Second Law seems cumbersome.

#### 3.2 Smooth transition to a large strain model

We have used a hypoplastic model\(^{19}\) to examine the problems that appear in the performance of PE in combination with other constitutive models. The hypoplastic (HP) tangential stiffness applied during dragging and the paraelastic tangential stiffness at $d^{AR1}_{\text{span}} = d_L$ should be similar. By this similarity we avoid strong kinks in stress-strain curves upon overloading of the $d_L$ circle. Due to directional dependence of both stiffnesses, a requirement of smooth transition for all directions of loading would be too restrictive. Let us assume, for simplicity $\nu = 0$ ($\lambda = 0$). Let the HP stiffness be $\mathbf{E}^{HP} = \frac{2\mu}{m_R} \mathbf{I}$, that is $m_R$ - times smaller than the largest PE stiffness. Moreover, we assume that $\mathbf{E}^{HP}$ remains constant for all strain spans $\mathbf{e}$ in the root circle and for all directions of loading $\mathbf{e}$. Depending on the span $\mathbf{e}$ and on the loading direction $\mathbf{e}$ the paraelastic stiffness given by (13) varies between $2\mu(1 - (1 + \chi) f d^{X}_{\text{AR}})$ for $\mathbf{e} \parallel \mathbf{e}$ and $2\mu(1 - f d^{X}_{\text{AR}})$ for all perpendicular directions. We have assumed $\nu = 0$, hence the stiffness upon unloading $\mathbf{E} = 2\mu \mathbf{I}$ holds.

The PE parameter $f$ should be chosen in accordance with $m_R$ in such way that the HP stiffness lies between the extreme values of the PE stiffness, namely

$$(1 + \chi)f \geq \frac{m_R - 1}{m_R d^{AR1}_{L}} \geq f \quad \text{(14)}$$

#### 3.3 Properties of the 1D hysteretic cycle

##### 3.3.1 Peak stress

In the 1D version our paraelastic model simplifies to a scalar $\sigma - \epsilon$ constitutive relation with the definition of distance $d^{AR1}_{\text{span}} = |\epsilon - \epsilon^{R1}| = |\epsilon|$ and with the scalar spans $\epsilon = \epsilon - \epsilon^{R1}$ and $s = \sigma - \sigma^{R1}$ measured from the latest reversal $R_1$. The constitutive relation (11) takes the form

$$s = (1 - f |\epsilon|^\chi) \mathbf{E} \epsilon \quad \text{with} \quad d^{AR1}_{\text{span}} = |\epsilon|$$

\(^{10}\) In this case the subroutines SDVINI and SIGINI may use the procedures of the main UMAT routine in Abaqus.

\(^{11}\) The elastic moduli depend on the stress and on the void ratio corresponding to the center of the root circle. These values may change only due to dragging and hence the moduli may be considered as material constants within the paraelastic model.
The material constants $E, f$ and $\chi < 1$ are all positive. In the case $e > 0$ and $\dot{e} > 0$ we have $|e| = e$ and differentiating $s = E e - f E \chi e^x$ with respect to $e$ we obtain the tangential stiffness
\[ \frac{ds}{de} = \frac{dx}{de} = E - (\chi + 1) f E |e|^\chi \] (16)
For $e < 0$ and $\dot{e} < 0$ we substitute $e = -\eta$ with $|e| = \eta$ into (15) and using the chain rule $ds/de = -ds/d\eta$ we arrive at (16) too. The tangential stiffness (16) should be always positive. The condition $ds/de > 0$ implies a limitation
\[ d_L < [(\chi + 1)\eta]^{-1/\chi} \] (17)
which is less restrictive than (14).

### 3.3.2 Damping and stiffness dependence on amplitude

The damping ratio $D$ is defined\(^{12}\) using the area $A = A_L + A_U - A_T$, Fig. 7, enclosed within the hysteretic cycle and normalized with stress and strain amplitudes
\[ D = \frac{A_L + A_U - A_T}{4\pi \frac{s}{E} \delta m_{\text{amp}} \chi} \] (18)
The area enclosed within the hysteretic cycle can be found from the analytical integration of the constitutive model.

![Fig. 7 The dissipated work upon a closed strain cycle. The abbreviation $F = f E$ is introduced.](image)

Using the notation introduced in Fig. 7 we derive an expression for the work upon a strain cycle from (15).

The energy supply $A_L$ for $de > 0$ in the range from $e = 0$ to $e = a$ is $2\epsilon_{\text{amp}}$ is
\[ A_L = \int_0^a E (e - \dot{e} f E |e|^\chi) de = \frac{1}{2} E a^2 \chi^2 - \frac{1}{2} f E a^2 \chi \]
The complementary energy for $de < 0$ is identical $A_U = A_L$ and using
\[ A_T = a (1 - f a \chi) E a \] (19)
we obtain
\[ A_L + A_U - A_T = \frac{\chi f E a^2 + \chi}{2 + \chi} \] (20)
The damping ratio can be calculated for any $\chi$ as:
\[ D = \frac{A_L + A_U - A_T}{4\pi \frac{s}{E} \delta m_{\text{amp}} \chi} = \frac{2f \chi a^{1+\chi}}{(2 + \chi) [a - f a^{1+\chi}]} \] (21)
The secant stiffness, evaluated at $\epsilon = a$ and for any value of $\chi$ is:
\[ H = \frac{s}{e} = E (1 - f a \chi) \] (22)

Realistic curves of modulus degradation and damping ratio vs. shear amplitude obtained with PE are shown in Fig. 8.

The material parameters $\chi, f$ can be determined from experimental results. Firstly, the ratio $m_R$ between secant PE stiffness $H$ for two different strain amplitudes must be defined. One of the selected strain amplitudes should coincide with the largest PE stiffness (plateau on the modulus degradation curve), Fig. 9, top.

The ratio $m_R$ is proposed to be determined from (22) in the range of amplitudes between $e = 0.01d_L$ and $10d_L$ with $d_L = 10^{-4}$, namely
\[ m_R = \frac{1 - f 10^{-2} d_L}{1 - f 10^{4} d_L^2} \] (23)
with $m_R \approx 5.0$ according to [7].

For a given $m_R, d_L$ and $D(d_L)$, e.g. from Fig. 9 we may numerically solve (23) and (21) for $f$ and $\chi$.

Fig. 8 shows the variation of the PE secant stiffness $H$ and of the damping ratio $D$ for different values of $m_R$. The curves are calibrated for $d_L = 0.2\%e$ and $D(d_L) = 0.02$ taken from Fig. 9, bottom. The stiffness ratio $m_R$ is calculated between strain amplitudes $2 \times 10^{-6}$ and $2 \times 10^{-3}$.

These values coincide with experimental curves [4, 7, 15, 18] for a clean sand, Fig. 9. A residual value of $D \approx 0.005$ for vanishingly small strain amplitudes is reported in [18]. We decided to disregard this effect here. This small residual value of damping may be caused by an experimental device. We simply shift the experimental damping curve by 0.005 downwards before calibration.

### 3.3.3 Masing rule

A sequence of cycles with gradually decreasing amplitude until it becomes vanishingly small is termed a shakedown. Let us examine the behavior upon a monotonic loading applied after a 1D shakedown.

According to the Masing rule [8] the back-bone curve $\sigma = F(\epsilon)$ should perfectly fit the unloading curve if scaled by two and rotated by $180^\circ$, that is
\[ \frac{\sigma - \sigma_R}{2} = F \left( \frac{\epsilon - \epsilon_R}{2} \right) \] (24)
wherein $\sigma^R$ and $\epsilon^R$ denote stress and strain at the most recent reversal point, and $F$ is the function describing the 1D relation between $\sigma$ and $\epsilon$ for the first loading. One can recognize the stress and strain spans appearing in (24).

Various propositions for $F()$ can be found in the literature, e.g. [2, 4, 13, 14], but their 3D generalizations are not presented.

The Masing rule is satisfied by the PE model if the number of reversal points during shakedown is sufficiently large, Fig. 10.

4 Continuity of stress

In the present section the 3D version of the PE model is discussed. The symmetry of the distance $d_{AR1} = d_{R1A}$ guarantees

$$s(\epsilon) = -s(-\epsilon)$$

(25)

for monotonous loading and monotonous unloading (only a single reversal point). The stress span, which has been built up upon loading, vanishes completely upon closing of the strain cycle independently of the strain path. Note that $s$ is a function of the current distance $d_{AR1}$ and the current strain span $\epsilon$ only and therefore our result holds despite possible intermediate (=overloaded) loops, even if they were not closed. The reversibility of stress upon the outer closed cycle is preserved.

The stress $\sigma(\epsilon)$ should be a continuous tensorial function (without jumps) of strain. The continuity of stress must hold in general case. Therefore, we must examine this continuity of stress for push and pop.

4.1 Continuity of stress after push

Continuity of functional $\sigma(\epsilon)$ means that infinitesimally small changes in strain increment cannot cause finite changes in stress increment. This requirement should be satisfied by any pair of similar strain increments, i.e. $\Delta \epsilon_a \approx \Delta \epsilon_b$, should result in $\Delta \sigma_a \approx \Delta \sigma_b$. Let us
consider strain increments close to the neutral direction. Let \( \Delta \epsilon_a \) cause unloading \( d^\text{pr}_{AR1} < d^a_{AR1} \) so that \( \epsilon^n \) is pushed on the stack whereas \( \Delta \epsilon_b \) corresponds to loading with \( d^\text{pr}_{AR1} > d^a_{AR1} \) (in Fig. 11).

The increments must be almost neutral and hence the incremented strains \( \epsilon^a_{n+1} \) and \( \epsilon^b_{n+1} \) must lie very close to the circle passing through \( \epsilon^n \) and \( \epsilon^{R1} \). They are practically touching this circle from the inside and from the outside, respectively. In both cases, therefore, the stress is calculated using the same stiffness \( H \) corresponding to \( d^\text{pr}_{AR1} \approx d^a_{AR1} \approx d^a_{AR1} \). For unloading with \( \Delta \epsilon_a \) we have

\[
\sigma^{n+1}_a = \sigma^{R1} + H : \epsilon^{n+1}_a = \sigma^{R2} + H : (\epsilon^{R1} - \epsilon^{R2}) + H : (\epsilon^{n+1}_a - \epsilon^{R1}) \quad (26)
\]

For loading with \( \Delta \epsilon_b \) we obtain

\[
\sigma^{n+1}_b = \sigma^{R2} + H : (\epsilon^{n+1}_b - \epsilon^{R2}) \quad (27)
\]

In (28) the most recent reversal has been denoted as \( R_2 \) (instead of \( R_1 \)) to ease the comparison with (26). The stress response to both increments is indeed almost identical, because (26) and (28) are calculated using the same stiffness \( H \) (based on the common diameter \( d^\text{pr}_{AR1} \approx d^a_{AR1} \)). Using PE with the Euclidean definition of distance we would obtain much larger \( H \) in the last summand in (27) compared with \( H \) in (28) so the difference \( \sigma^{n+1}_a - \sigma^{n+1}_b \) could be substantial.

The continuity of stress response for different directions of stretching can be visualized using polar plots of stiffness, so called response envelopes or response polars [3]. The definition (2) of distance\(^{13} \) causes a rapid change in stiffness which is represented by the kinks in the response envelope for neutral strain directions (i.e. for \( N : \Delta \epsilon \approx 0 \), see the profiles in Fig. 12). Despite the kinks, the continuity of stress response across the neutral direction is satisfied which is of advantage compared to the definition proposed in \([5, 6]\). Similar concave kinks are often generated by elastoplasticity.

4.2 Continuity of stress after pop

The continuity of stress should also hold after a reversal is popped. Consider an infinitesimally small strain increment \( \Delta \epsilon \approx 0 \) that pops the reversal \( R_1 \), Fig. 13. This implies almost equal distances \( d^\text{pr}_{AR1} \approx d^a_{RR1} \approx d^a_{RR1} \) satisfying \( d^a_{AR1} < d^a_{RR1} < d^a_{RR1} \). The stress \( \sigma^n \) before pop can be calculated from \( R_2 \) as

\[
\sigma^n = \sigma^{R1} + H : \epsilon^n = \sigma^{R2} + H : (\epsilon^{R1} - \epsilon^{R2}) + H : (\epsilon^n - \epsilon^{R1}) \quad (29)
\]

with both stiffnesses \( H \) corresponding to \( d^a_{AR1} \approx d^a_{RR1} \) (Fig. 13, left). After the increment we pop \( R_1 \) but preserve the numbers of reversals, that is, we do not rename \( R_2 \) to \( R_1 \) (Fig. 13, right).

\[
\sigma^{n+1} = \sigma^{R2} + H : (\epsilon^{n+1} - \epsilon^{R2}) \quad (30)
\]

\[
\epsilon^{n+1} - \epsilon^{R2} = \epsilon^{n+1} - \epsilon^{R1} + \epsilon^{R1} - \epsilon^{R2}
\]

---

\(^{13}\) as the diameter of the circle connecting the reversal \( R_1 \) and the current state \( A \).
and with $H$ corresponding to $d_{AR1}^{n+1} \approx d_{R1}$. Comparing (29) and (30) for infinitesimally small increment, that is for $d_{AR1}^n \approx d_{AR1}^{n+1}$ and $\varepsilon^n \approx \varepsilon^{n+1}$ we conclude $\sigma^n \approx \sigma^{n+1}$ which means that the continuity requirement is satisfied.

5 Drift of stress paths

We examine the performance of the PE upon strain paths in the isomorphic $\varepsilon_P - \varepsilon_Q$ space. Let a monotonic strain path along $\varepsilon_P$ axis be preceded by a 1D shakedown along the $\varepsilon_P$ axis, Fig. 14.

Judging by the diagonal isotropic elastic $2 \times 2$ stiffness $E = \text{diag}(3K, 2G)$ one could expect that the PE response to $\varepsilon_Q$ straining is a stress path along the $Q$-axis. It turns out, however, that the stress path corresponding to the monotonic straining along $\varepsilon_Q$ has a zigzag form quantified with $\dot{s}_Q/\dot{s}_P = \pm z$. We have not found a remedy against it and to some extend (say, for $z > 4$) this zigzag must be accepted. Before a detailed discussion of the zigzag effect we should mention that a related phenomenon has been observed in laboratory [1, 16, 17], see Fig. 15. Straining along $\varepsilon_Q$ was preceded either by volumetric compression or by volumetric extension. Evidently, the stress path is slightly drifted off the vertical direction depending on the strain history.

Conversely, in order to follow a purely deviatoric stress path one needs a small volumetric compression or extension in experiment, depending of the sign of the preceding isotropic deformation [16], Fig. 16.

The drift of the stress path is caused by the $d_{AR1}$-dependent secant stiffness $H = (1 - f d_{AR1})E$ proposed in (11). The effect can be understood examining the tangential PE stiffness $K$

$$K = \frac{\partial \sigma}{\partial \varepsilon} = \frac{\partial s}{\partial \varepsilon} = H - \chi f d_{AR1}^{-1} \epsilon \eta$$

with (31)

$$\eta = \frac{\partial d_{AR1}}{\partial \varepsilon} = -2\psi \epsilon + \frac{N^{R1}}{\psi^2}$$

and $\psi = \epsilon : N^{R1}$ (32)

It can be shown that $\|\eta\| = 1/\psi^2$ and that $\eta$ is perpendicular to the loading circle. The distribution of $\eta$ on a loading circle is presented\(^{14}\) in Fig. 17. Note that $\lim_{A \to R} \|\eta\| = \infty$. The stress rate $\dot{s} = K : \dot{\varepsilon}$ consists of the basic portion $H : \dot{\varepsilon}$ and of the drift $-\chi d_{AR1}^{-1} \epsilon : \eta : \dot{\varepsilon}$ which is responsible for the characteristic drift of the stress path after $90^\circ$ turn and for the zigzag. The stress rate due to drift is proportional to $E : \eta$ whereas the direction of basic stress rate is proportional to $E : \dot{\varepsilon}$ like in conventional elasticity. The directions $E : \eta$ and $E : \dot{\varepsilon}$ can be very different, which explains the deflection of the PE stress path compared the elastic one. The drift is particularly strong for large $\|\eta\|$ with $\epsilon \parallel \eta$ and $\epsilon \perp \dot{\varepsilon}$.

Let us now compare the stress rates obtained for deviatoric strain rate from the two cases depicted in Fig. 18. Their difference is responsible for the zigzag in Fig. 14. In both cases we have almost the same $d_{AR1}$ and $\eta$. According to (31) the difference between

\(^{14}\) Mathematica script to plot the region $1/\|\eta\|$

```
b = \{(r, 0) + r (\cos[a], \sin[a])\};
ub = \{-1, 0\};
h = -\Re[eb \cdot ub];
\{solu\} = \text{Solve}\{h == 2, r[[1]]\};
\{psi\} = \text{Normalize}[eb / solu, ub];
eta = \{1/\Re[psi]^2\};
PolarPlot[{r / solu, r + 1/etanorm / solu}, \{a, -Pi, Pi\}]```

Fig. 14 The continuity of stress provokes changes in the curvature of the stress increments after a shakedown followed by consecutive pops.

Fig. 15 Influence of the former volumetric deformation on the shape of the stress curve obtained with paraelasticity upon deviatoric strain path.

Fig. 16 Strain vectors observed experimentally after the turning point O of the stress path [16].
stress rates in these two cases results from different drifts

\[(K^{\text{comp}} - K^{\text{ext}}) : \dot{\varepsilon} = -f_\chi d_{AR1}^{\chi-1} \mathbf{E} : (\varepsilon^{\text{comp}} - \varepsilon^{\text{ext}}) (\eta : \dot{\varepsilon})\]

The inclination of the stress path during an increase of \(e_Q\) at \(e_p = \text{const}\) can be quantified using the PE equation in the P-Q space with

\[
\begin{pmatrix}
  s_P \\
  s_Q
\end{pmatrix} = (1 - f d_{AR1}^\chi) \begin{pmatrix} 3K & 0 \\
  0 & 2G
\end{pmatrix} \begin{pmatrix} e_p \\
  e_Q
\end{pmatrix}
\]

and the stress rate due to \(\dot{e}_Q\) is

\[
\begin{pmatrix}
  \dot{s}_P \\
  \dot{s}_Q
\end{pmatrix} = -\chi f d_{AR1}^{\chi-1} d_{AR1}^\chi \dot{e}_Q \begin{pmatrix} 3K e_p \\
  0
\end{pmatrix}
+ \frac{(1 - f d_{AR1}^\chi)}{2G} \dot{e}_Q\]

wherein \(d_{AR1} = \frac{\partial d_{AR1}}{\partial e}\).

The drift of the stress path can be quantified and bounded as a function of the PE constants. For the case of an isotropic compression followed by purely deviatoric shearing, the drift \(\dot{s}_P/\dot{s}_Q\) is obtained as the ratio

\[
\frac{\dot{s}_P}{\dot{s}_Q} = \frac{-\chi f d_{AR1}^{\chi-1} d_{AR1}^\chi 3K e_p}{-\chi f d_{AR1}^{\chi-1} d_{AR1}^\chi 2G e_Q + (1 - f d_{AR1}^\chi) 2G}
\]

or

\[
\frac{\dot{s}_P}{\dot{s}_Q} = \frac{3K e_p}{2G e_Q + (1 - \frac{1}{d_{AR1}^\chi}) 2G}
\]

Note that for a given \(e_Q\) the sign of inclination \(\dot{s}_P/\dot{s}_Q\) depends on the sign of \(e_p\) so switching \(e_p\) due to consecutive pops causes the zigzag shown in Fig. 14. For the special case \(\nu = 0\) we have \(3K = 2G\) and for \(e_p \approx e_Q \approx d_{AR1}/2\) with \(d_{AR1} \approx 2\) we obtain an estimation

\[
\frac{1}{z} = \frac{\dot{s}_P}{\dot{s}_Q} = \frac{1}{1 + \frac{1}{\chi} \left(1 - \frac{1}{d_{AR1}^\chi}\right)}
\]

with the inclination parameter, say \(z > 4\), (see Fig. 19). This condition defines a rough restriction for the model

\[
d_L < \left[f \left(z + 1 + \frac{1}{\chi}\right)\right]^{-1/\chi} \approx 0.3\%
\]

assuming \(\chi \approx 0.9\) and \(f \approx 300.0\). Other, possibly more restrictive conditions to define the range of applicability of the proposed paraelastic model are shown in Appendix 7.

\[
\dot{\mathbf{d}}_{AR1} = ||\mathbf{e}||
\]

which is simpler than our definition of \(d_{AR1}\) given in (2). They proposed constitutive equations that described the compliance rather than stiffness and did not include dilatancy, cf. [11], but the general approach with spans of strain and stress was similar. However, the Euclidean distance \(\mathbf{d}_{AR1}\) caused a serious problem of discontinuity in functional \(\boldsymbol{\sigma}(\varepsilon)\). Two examples are illustrated in Fig. 20.

Let us first consider two strain increments applied along the loading circle \(\mathbf{d}_{AR1} = \text{const}\) (Fig. 20, left).

Fig. 17 Distribution of \(\eta\) on a loading circle. Note that \(1/||\eta||\) and not infinite \(||\eta||\) is plotted.

Fig. 18 LEFT: deviatoric strain path (ext) preceded by isotropic extension. RIGHT: deviatoric strain path (comp) preceded by isotropic compression. The volumetric deformation has too strong impact on the tangential stiffness for large distances \(d_{AR1} \approx d_L\).

Fig. 19 The slope of an increment in the stress span \(\dot{s}_P/\dot{s}_Q\) is quantified as a function of \(d_{AR1}, f, \chi\). A reduced value of \(d_{AR1}\) can be chosen to reduce the zigzag effect.

6 PE with Euclidean distance

In the seventies Hueckel and Nova [5, 6, 12] proposed a paraelastic model based on the Euclidean distance

\[
\mathbf{d}_{AR1} = ||\mathbf{e}||
\]
Two strain increments $Δε_a$ and $Δε_b$ have been chosen almost perpendicularly to $e$ so that they lie infinitesimally close to loading circle (depicted as the arc in Fig. 20). The increment $Δε_a$ corresponds to loading, and the increment $Δε_b$ to unloading with a small $d_{AR1}$ calculated from the new reversal point. Hence the stiffness in the latter case is much larger and in consequence $Δσ_a$ is greater than $Δσ_b$, which can be seen as discontinuity in functional $σ(e)$. The problem disappears if we use $d_{AR1}$ instead of $d_{AR1}$.

Another discontinuity in $σ(e)$ may appear along a monotonous strain path (vertical line) interrupted by a small unloading, Fig. 20, right. If this unloading occurred vertically downwards and were followed by reloading and overloading also along the vertical line the whole event would be wiped out from the material memory. However, we may generate a reversal in the form of a small spiral section constructed in such way that the Euclidean distance $d_{AR1} = ||e||$ upon the spiral is monotonically growing. Along such path we may return to the original direction of deformation and then continue along the vertical line. An infinitesimally small spiral disturbance may therefore lead to the update of the reversal point and to the substantial change in the stiffness, which can be seen as discontinuity in functional $σ(e)$. Again, the problem disappears if we use $d_{AR1}$ instead of $d_{AR1}$.

### 7 Outlook

The presented model has been extended incorporating the description of reversible dilatancy-contractancy effects. This extension as well as the finite element implementation of the extended paraelasticity are presented in a companion paper [11].

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### A. Determination of the split parameter $θ$

The parameter $θ$ is defined as

$$θ_{1,2} = \frac{-b ± \sqrt{b^2 - 4ac}}{2a}$$

with

$$a = Δε · Δε$$

$$b = 2r · Δε$$

$$c = r : r - \frac{d^2}{4}$$

is the solution to the quadratic form

$$\frac{d^2}{4} = r (r + θΔε) (r + θΔε)$$

$$r = e^n - \left(\frac{d^2}{2} - N^2\right)$$

The valid solution is $max(θ_1, θ_2, 0)$.

### B. Push, pop and split procedures in space P-Q

The following MATHEMATICA script illustrates how the push, pop and split procedures can be implemented (here for the isomorphic P-Q space)

```mathematica
push[ An_, An1_, R1_ ] := Module[ {\[Epsilon], \[Sigma], N0, d0, epor0, aP1, \[Sigma]1, \[Epsilon]1, N1, d1 , epor1, cP1, koniec},
  If[Length[state] == 3 && state[[2, 4]] ~approx~ state[[3, 4]],
    state = Delete[state, 3]; (* from two reversals on the max circle take the most recent one as root *)
    completePEStress[ ]; ]; ];

pop[ ] := Module[ {\[Epsilon], \[Sigma], N0, d0, epor0, aP1, \[Sigma]1, \[Epsilon]1, N1, d1 , epor1, cP1, koniec},
  If[Length[state] <= 2, Print[ "POP: Dragging "] ; Goto[koniec] ; ];
  state = Prepend[state, An1];
  If[Length[state] == 3 && state[[2, 4]] ~approx~ state[[3, 4]],
    state = Delete[state, 3]; (* from two reversals on the max circle take the most recent one as root *)
    completePEStress[ ]; ];
  If[state[[2, 4]]] ~smallerApprox~ state[[1, 4]],
    pop[ ]; (* recursive call of pop *)
  Label[koniec]; ];

split[ An_, An1_, R1_ ] := Module[{dBefore, dAfter, dl, \[Epsilon]1, n1, \[Epsilon]0, \[Epsilon]nl, \[Epsilon]n0, \[Epsilon]-n0, \[Epsilon]-nl, A, an, ab, ha, de, d, rb, a, b, c, discr, PFPortion, \[Epsilon]1, \[Epsilon]n, SplitExit, depsPE,depsPEdrag},
  (* splits the increment in a paraelastic portion, updates stress and returns the protruding portion of the increment which has to be dragged *)
  dBefore = (An[4], An1[4], R1[4]); (* dBefore *)
  \[Epsilon]1 = \[Epsilon][1]; \[Epsilon]nl = \[Epsilon][n1]; \[Epsilon]-n0 = \[Epsilon][-n0]; \[Epsilon]-nl = \[Epsilon][-nl];
  an = \[Epsilon][1];
  ab = \[Epsilon][0];
  ha = \[Epsilon]0 - \[Epsilon]-n0; d = An[4];
  deps = \[Epsilon]-n0 - \[Epsilon]-nl; rb = \[Epsilon]-nl - \[Epsilon]-n0; dl2 = dl/2 ;
```
c = rb.rb - dL^2/4;
b = 2*rb.deps;
a = deps.deps;
If[a<approx>0, PEPortion = 1; depsPE = 0*deps; depsDrag = 0*deps;
Goto[SplitExit];
discr = b^2 - 4*a*c;
If[discr < 0, Print["Error in Split"]]; abort[];
discr = Sqrt[discr];
Goto[SplitExit];
}

The complete source code of the Mathematica script is available from the second author.

C. Validity range
The usual range of application of the PE is 2<sup>amplo</sup> < d<sub>L</sub> ≈ 0.2%o. However, for the completeness of the formulation we analyse the hypothetical case for which σ(e) given by (11) is not invertible, 2<sup>amplo</sup> ≈ 0.2%o. We are interested in a rough estimation of the invertibility limit only. In order to preserve the 1-1 dependence between e and s for loading, we require that the s − e curve should be monotonically increasing independent of the direction of e. For this purpose the condition of the positive tangential stiffness should be satisfied

∀e : s = e : (1 - d<sub>AR1</sub>)E - fχd<sub>AR1</sub>-1E : e η ≥ 0 (46)

meaning that all eigenvalues of the tangent stiffness are positive. The invertibility limit corresponds to det s′ = 0. We factorize (47)

s′ = (1 - d<sub>AR1</sub>)E - fχd<sub>AR1</sub>-1E : e η

and since E is positive definite, the singularity of s′ results from the expression in brackets. Using the algebraic relation det(A + a) = det(A)(1 + a : A : -1 : a) we obtain

det[(1 - d<sub>AR1</sub>)E - fχd<sub>AR1</sub>-1E : e η] = 0

and using η : J : e = -∥e∥/ψ = d<sub>AR1</sub>

det[(1 - d<sub>AR1</sub>)J] = 0

from which follows the maximum double amplitude

d<sub>AR1</sub> < min{ f<sup>-1</sup>∥, [(1 + χ)f]<sup>-1</sup>∥} = [(1 + χ)f]<sup>-1</sup>∥

Alternatively, one may postulate that the Euclidean norm of strain path must increase monotonously with d<sub>AR1</sub>. (s : s) > 0 at d<sub>AR1</sub> > 0. This condition is somewhat more restrictive than the positiveness of s′.

For a rough evaluation of the limiting double amplitude we consider a simple case E = E1 and e = 0 for which

s : s = (1 - d<sub>AR1</sub>)E^2 with u = e : e

and

∂s : s = 2(1 - fχd<sub>AR1</sub>-1)(-χfχd<sub>AR1</sub>-1)uE^2

+ (1 - d<sub>AR1</sub>)v^2 χd<sub>AR1</sub> E^2 > 0

wherein

∂u : s = -e : N d<sub>AR1</sub>^2 = u / d<sub>AR1</sub>

Substituting the latter expression in (54) we find

∂s : s = uE^2(1 - fχd<sub>AR1</sub>-1)2(1 - χfχd<sub>AR1</sub>-1)AR1

+ (1 - d<sub>AR1</sub>)v^2 1 / d<sub>AR1</sub>^2 > 0

and finally

d<sub>AR1</sub> < [f(1 + 2χ)]<sup>-1</sup>χ

D. List of Symbols
The following symbols are used in this paper:

<sup>1</sup> I = diag(1, 1, 1)
<sup>1+1</sup> I = diag(2, 2, 1)

A 1D double strain amplitude

A<sub>L</sub>, A<sub>U</sub>, A<sub>T</sub> Area of a loop s − e

A<sub>n</sub> State before the current increment

A<sub>n+1</sub> State after the current increment

d<sub>AR1</sub> Diameter of the circle passing through e<sup>A</sup> and e<sup>Ri</sup>

d<sub>L</sub> Diameter of the largest PE circle (material parameter)

d<sub>AR1</sub>, d<sub>i</sub> Rate of the diameter of the loading circle

d<sub>Ri</sub> Diameter of the circle passing through e<sup>Ri</sup> and e<sup>Ri+1</sup>

D<sub>e</sub>, D<sub>e</sub> Void ratio at the th reversal R<sub>i</sub> and at the current state A<sub>n</sub>, respectively

e 1D strain span

ε, ε<sup>Q</sup> Isomorphic invariants ε<sub>P</sub>, ε<sub>Q</sub>

η Isomorphic invariants of the strain span e

E 1D elastic stiffness

E Isotropic hypoelastic tangential stiffness

f material constant for PE stiffness

G Elastic shear modulus

H 1D PE secant stiffness

i Index of reversals

l Fourth order identity tensor
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K$</td>
<td>Elastic bulk modulus</td>
</tr>
<tr>
<td>$l$</td>
<td>Length of the stack, $l \geq 2$</td>
</tr>
<tr>
<td>$L$</td>
<td>State at the last indelible reversal described by $L = {\varepsilon^L_1, \varepsilon^L_2, N^L_1, d^L_1, \varepsilon^L, P_{RiL}}$</td>
</tr>
<tr>
<td>$m_R$</td>
<td>Ratio between the largest PE secant stiffness and the secant stiffness for a larger strain amplitude</td>
</tr>
<tr>
<td>$N^R_i$</td>
<td>Outer normal to the reversal circle at $\varepsilon^R_i$</td>
</tr>
<tr>
<td>$P$</td>
<td>Isomorphic pressure $P = -\frac{1}{\nu}$</td>
</tr>
<tr>
<td>$P_{ARi}$</td>
<td>Reversible dilatancy/contractancy effect manifested in terms of pressure</td>
</tr>
<tr>
<td>$Q$</td>
<td>Isomorphic invariant of stress deviator $Q = |\sigma^A|$</td>
</tr>
<tr>
<td>$\bar{Q}$</td>
<td>Isomorphic PE elastic coefficient $P/K \geq 0$. Diameter of the largest PE circle.</td>
</tr>
<tr>
<td>$\nu$</td>
<td>PE Poisson number.</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>Isomorphic PE elastic coefficient $P/K = 0.001$. Isomorphic PE elastic coefficient (Butterfield).</td>
</tr>
<tr>
<td>$\nu_{PE}$</td>
<td>$\nu_{PE} = 0.2$. PE Poisson number.</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>1D shear strain amplitude</td>
</tr>
<tr>
<td>$\bar{\gamma}$</td>
<td>Fraction of the strain increment within the largest PE circle before dragging</td>
</tr>
<tr>
<td>$\vartheta$</td>
<td>Outer normal to the loading circle</td>
</tr>
<tr>
<td>$\lambda, \mu$</td>
<td>Lamé parameters</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Outer normal to the loading circle</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>1D stress (compression negative)</td>
</tr>
<tr>
<td>$\bar{\sigma}$</td>
<td>Stress (tension positive)</td>
</tr>
<tr>
<td>$\tau$</td>
<td>1D shear stress</td>
</tr>
<tr>
<td>$\tau^R$</td>
<td>1D shear stress at the most recent reversal</td>
</tr>
</tbody>
</table>

Material parameters

The following is the list of material parameters required by PE. Some intuitive values are suggested.

- $\kappa_{PEE0} = P/K = 0.001$. Isomorphic PE elastic coefficient (Butterfield).
- $\nu = \nu_{PE} = 0.2$. PE Poisson number.
- $f = 304.57$.
- $\chi = 0.956$. Exponent ($< 1$) controlling the damping ratio.
- $d_L = 0.0002$. Diameter of the largest PE circle.

References