High-Cycle Accumulation Model with EAS Elements

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Abstract. A FE implementation in the commercial program ABAQUS\textsuperscript{®} of the high cycle accumulation model for sand is discussed. Due to the large gradients of the predicted accumulation, elements with enhanced assumed strains are used. Application of EAS to constitutive models formulated in rates (incrementally nonlinear and path dependent) requires, apart from return mapping and equilibrium iterations, an additional iterative process for determination of the enhanced strain. It is demonstrated that EAS elements can cope well with difficult accumulation fields and with the strong material nonlinearity as required by the HCA model.

Keywords. high cycle accumulation, locking, enhanced assumed strains, hypoplasticity, material nonlinearity

1. Introduction

The high cycle accumulation (HCA) model \([5]\) is an unusual constitutive description of sand. Two kinds of loading are handled by the HCA model: the monotonic deformation and the high cycle loading. The response to monotonic deformation is elasto-plastic (conventional), whereas the response to fatigue loads\(^2\) is the trend of stress accumulation (relaxation). The HCA model has been intensively used for the last ten years and several improvements have been made. Finite element (FE) calculations revealed two problems connected with the HCA: the excessive self-stress due to locking and poor predictions in combinations of cyclic and monotonic loading. Locking can only partly be eliminated by the so called reduced spatial integration. A more flexible technique with the so called enhanced assumed strains (EAS) is discussed in this paper. The first attempt to formulate an EAS element (in the form of a MATHEMATICA package) for the HCA model \([3]\) was limited to linear elasticity with the initial strain (due to accumulation). This method was extended for incrementally linear HCA models and implemented to ABAQUS\textsuperscript{®} in \([2]\). In order to handle the incrementally nonlinear version of HCA as proposed in \([4]\) further extensions were necessary.

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\(^2\)a single load increment consists of many cycles with a constant strain amplitude
1.1. FE Implementation of the High Cycle Accumulation (HCA) Model

According to the high cycle constitutive model \[5\] (HCA) for soils

\[
\varepsilon_{ij} = E_{ijkl}(\dot{\varepsilon}_{kl} - \dot{\varepsilon}_{acc}^k - \dot{\varepsilon}_{pl}^k)
\]  (1)

the rate of accumulation \(\dot{\varepsilon}_{acc}\) may strongly vary in space, approximately proportional to the square of the amplitude\(^3\). For example, in a typical punch problem of a shallow foundation the stress amplitude caused by a vertical point load \(P_{ampl}\) on the surface of the elastic half-space can be roughly estimated using the well known formulas by Boussinesq (3D case) or by Flamant (for line load \(\tilde{P}_{ampl}\), 2D case). For line loads \(\varepsilon_{ampl} \sim \sigma_{ampl}\) decreases linearly in space and hence \(\dot{\varepsilon}_{acc}^i j\) decreases with the square distance \(r^2\) from the foundation in plane strain problems. Plane strain finite elements used with HCA should be able to reproduce at least all strain fields which vary proportionally to \(r^2\).

This requirement may be very important for the quality of the FE-results with HCA because overly constrained elements response with artificial (numerical) self-stresses. Such stresses may spoil the subsequent calculation of accumulation. The feed back phenomenon causes that the numerical self-stress amplifies itself. In order to avoid this problem we use the enhanced assumed strain (EAS) elements which introduce additional strain fields \(\tilde{\varepsilon}_{ij}\) rendering the element more flexible.

2. Three Field EAS Equations

The essential assumption of the three field approach is the form of strain

\[
\varepsilon_{ij} = u_{(i,j)} + \tilde{\varepsilon}_{ij}
\]  (2)

consisting of the displacement compatible part \(u_{(i,j)}\) and the enhancement \(\tilde{\varepsilon}_{ij}\). Given the internal energy density \(\psi(\varepsilon_{ij})\) and considering \(\delta u\) and \(\tilde{\varepsilon}_{ij}\) as primary fields the variation of the potential energy \(\Pi\) the variation \(\delta \Pi\) should vanish

\[
\delta \Pi = \int \sigma_{ijkl}^f(\delta \tilde{\varepsilon}_{ij} + \delta u_{(i,j)})dV - \int_{t_0}^{t_1} \delta u_t dS - \int f_i(\delta u_i dV
\]

\[
- \int \sigma_{ij} \delta \tilde{\varepsilon}_{ij} dV - \int \sigma_{ij} \delta \tilde{\varepsilon}_{ij} dV = 0
\]  (3)

Collecting terms at independent variations \(\delta u_i, \delta \tilde{\varepsilon}_{ij}\) and \(\delta \sigma_{ij}\) we obtain

\[
\begin{cases}
\int \delta u_{(i,j)} \sigma_{ij}^f dV = \int_{S_0} \delta u_t dS + \int f_i \delta u_i dV \\
\int \delta \tilde{\varepsilon}_{ij}(\sigma_{ij}^f - \sigma_{ij})dV = 0 \\
\int \delta \sigma_{ij} \tilde{\varepsilon}_{ij} dV = 0
\end{cases}
\]  (4)

The assumption of EAS is, beside (2), the orthogonality of stress and enhanced strain enforced by specially designed interpolation functions.

\(^3\)Recent triaxial tests [8] show that the exponent \(n\) in \(\varepsilon_{acc}^i j \sim (\varepsilon_{ampl})^n\) can be smaller than two, typically \(1.4 < n < 2.0\). However, some additional spatial variability of \(\varepsilon_{acc}^i j\) results from the barotropic stiffness (deformation amplitude is larger at the ground surface), from the stress obliquity (in the vicinity of the foundation the accumulation is larger due to larger mobilized friction angle) and from the barotropy of the accumulation (\(\varepsilon_{acc}^i j\) is larger at low effective mean stress, i.e. at the ground surface).
3. Incremental Equilibrium in EAS Element with Material Nonlinearity

If a constitutive model does not allow for a formulation of \( \psi(\varepsilon_{ij}) \), we start the three-field formulation from the strong form of static equilibrium \( \sigma_{ij,j} + f_i = 0 \) from kinematic strain - displacement relation \( \varepsilon_{ij} - u_{(i,j)} = 0 \) and from the compatibility condition \( \sigma_{ij} - \sigma_{ij}^e = 0 \) of the constitutive stress \( \sigma_{ij}^e \) and the master stress field \( \sigma_{ij} \). After the usual integration by parts of the static equilibrium we obtain the weak forms. In nonlinear problems (without \( \psi(\varepsilon_{ij}) \)) the incremental form is used

\[
\begin{align*}
\int \delta u_{(i,j)} \hat{\sigma}_{ij}^e dV &= \int_S \delta u_{t,i} dS + \int \delta u_{,i} dV - \int \delta u_{(i,j)} \sigma_{ij}^e dV \\
\int \delta \hat{\varepsilon}_{ij} (\hat{\sigma}_{ij} - \hat{\sigma}_{ij}) dV &= - \int \delta \hat{\varepsilon}_{ij} (\sigma_{ij}^e - \sigma^e_{ij}) dV \\
\int \delta \hat{\sigma}_{ij} \hat{\varepsilon}_{ij} dV &= - \delta \hat{\sigma}_{ij} \hat{\varepsilon}_{ij}^e dV,
\end{align*}
\]

wherein the stress increment is found from nonlinear constitutive function \( \sigma_{ij} = \sigma_{ij}^e - \sigma_{ij}^e = f_{ij}(\varepsilon_{ij}) \). The interpolation functions are specially chosen in such form that (5) is satisfied independently of the coefficients of discretization. Moreover, these interpolation functions render also \( \int \hat{\sigma}_{ij} \delta \hat{\varepsilon}_{ij} = 0 \) and \( \int \sigma_{ij}^e \delta \hat{\varepsilon}_{ij} = 0 \) in (5)\(_2\). Hence (5) can be reduced to

\[
\begin{align*}
\int \delta u_{(i,j)} \hat{\sigma}_{ij}^e dV &= \int_S \delta u_{t,i} dS + \int \delta u_{,i} dV - \int \delta u_{(i,j)} \sigma_{ij}^e dV \\
\int \delta \hat{\varepsilon}_{ij} \hat{\sigma}_{ij} dV &= 0.
\end{align*}
\]

These equations must be satisfied at the end of each load increment. The equilibrium is reached in the course of the usual equilibrium iteration (EI) in which the nodal displacements are refined.

3.1. Iterative Determination of \( \hat{\varepsilon}_{kl} \)

Let us consider a single equilibrium iteration within a single EAS element. Our element is given just \( \hat{\varepsilon}_{kl} \) and we need to calculate the best enhancement \( \hat{\varepsilon}_{ij} \) before the subsequent EI begins. The following discretization is introduced

\[
\hat{u}_{,i} = N_{0i} \hat{u}_{0i}, \quad \hat{u}_{,i,j} = N_{0i,j} \hat{u}_{0i,j}, \quad \hat{\varepsilon}_{ij} = M_{ij\alpha} \hat{\varepsilon}_{\alpha} \quad \text{and} \quad \hat{\sigma}_{ij} = P_{ij\beta} \hat{\sigma}_{\beta}.
\]

We substitute \( \delta \hat{\varepsilon}_{ij} = M_{ij\alpha} \delta \hat{\varepsilon}_{\alpha} \) into (6)\(_2\) obtaining

\[
\int M_{ij\alpha} \hat{\sigma}_{ij}^e dV = 0\alpha
\]

\( ^4 \)The hyperelasticity \( E_{ijkl} \hat{\varepsilon}_{kl} \) is incrementally linear. However hypoplasticity \( E_{ijkl} \hat{\varepsilon}_{kl} = L_{ijkl} \hat{\varepsilon}_{kl} + N_{ijkl} (\hat{\varepsilon}_{kl} \hat{\varepsilon}_{kl}) \) or elastoplasticity \( E_{ijkl} \hat{\varepsilon}_{kl} = E_{ijkl} \hat{\varepsilon}_{kl} - \frac{E_{ijkl} \hat{\varepsilon}_{kl}}{h_{ijkl} \varepsilon_{ijkl}^{\text{acc}}} \) or HCA \( E_{ijkl} \hat{\varepsilon}_{kl} = E_{ijkl}^{\alpha} \hat{\varepsilon}_{kl} - \varepsilon_{ijkl}^{\text{acc}} \) are incrementally nonlinear. We avoid the form \( \hat{\sigma}_{ij} = E_{ijkl} \hat{\varepsilon}_{kl} \) with the secant stiffness \( E_{ijkl}^{\alpha} \) which may itself depend on the strain increment.
The stress increment $\Delta\hat{\sigma}_{ij}$ is a nonlinear function of the enhanced strain $\hat{\varepsilon}_{kl} = \hat{u}_{(k,l)} + \hat{\varepsilon}_{kl}$. In order to solve (8) for $\hat{\varepsilon}_\beta$ we calculate iteratively the corrections $\hat{\varepsilon}_\beta$ with the Newton scheme $(\partial\sigma/\partial\varepsilon)\hat{\varepsilon}_\beta = -\hat{r}_\alpha$, wherein $\hat{r}_\alpha$ denotes the residuum of (8). Given a fixed value $\hat{u}_{(k,l)}$ and a predictor of $\hat{\varepsilon}_{kl}$ we calculate $\hat{\sigma}_{ij}$ and $E_{ijkl} = \partial\hat{\sigma}_{ij}/\partial\hat{\varepsilon}_{kl}$ conventionally$^5$ and then we solve$^5$

$$
\left[ \int M_{ij\alpha}E_{ijkl}M_{kl\beta}dV \right] \hat{\varepsilon}_\beta = -\int M_{ij\alpha}\hat{\sigma}_{ij}dV \quad \text{with} \quad M_{\beta ab} = \frac{\partial\hat{\varepsilon}_{ab}}{\partial\hat{\varepsilon}_\beta} \quad (9)
$$

for the correction $\hat{\varepsilon}_\beta$. Finally the new $\hat{\varepsilon}_{kl} = M_{kl\beta}(\hat{\varepsilon}_\beta + \hat{\varepsilon}_\beta)$ is used to refine $\hat{\sigma}_{ij}$. Calculation of $\hat{\sigma}_{ij}^P$ is accomplished by invoking the constitutive routine $\text{umat}$ and may need a return mapping iteration inside $\text{umat}$, if an implicit time integration is used. After the above iteration is completed we obtain the exact value of $\hat{\varepsilon}_\beta$ and the corresponding $\hat{\varepsilon}_{kl}$.

3.2. Equilibrium Iteration

For the equilibrium equation we need the element stiffness matrix and the element contributions to the loading vector. Given a predictor $\hat{u}_i$ of displacement$^7$ the discretized form of (6) is in general not satisfied$^8$

$$
\begin{align*}
\int N_{a j} \left( \sigma_{ij}^P \right) dV - \int S_{kl} \varepsilon_{kl} dS - \int N_{a i} f_i dV &= r_{a i} \\
\int M_{ij\alpha} \left( \sigma_{ij}^P \right) dV &= 0
\end{align*} \quad (10)
$$

The residual $r_{a i}$ should vanish after a suitable correction $c_{il}^P$ added to $\hat{u}_i$. We apply the Newton scheme $(\partial\tau/\partial\varepsilon)c = -r$ to (10)$^1$. Let us assume a minor symmetry of $E_{ijrs} = \partial\sigma_{ij}/\partial\varepsilon_{rs}$, i.e. $E_{ijrs} = E_{ijsr}$, which allows for simplification $E_{ijrs}\hat{u}_{l(i,j)} = E_{ijsr}\hat{u}_{l(j,i)}$. Using the discretized form of strain with $\partial u_{rs}/\partial u_{lk} = N_{rs},\delta_{lk}$ and with $\partial\varepsilon_{kl}/\partial\varepsilon_{kl} = M_{kl\beta}$ we obtain (10) in the algebraic form$^9$

$$
\begin{align*}
\left[ \int N_{a j} E_{ijkl}N_{r,kl} dV \right] c_{il}^P + \left[ \int N_{a j} E_{ijrs}M_{rs\beta} dV \right] c_{il}^P &= -r_{a i} \\
\left[ \int M_{ij\alpha}E_{ijkl}N_{r,kl} dV \right] c_{il}^P + \left[ \int M_{ij\alpha}E_{ijrs}M_{rs\beta} dV \right] c_{il}^P &= 0
\end{align*} \quad (11)
$$

It is convenient to write the short form of (11)

$^5$The constitutive stress increment is calculated from the predictor of strain including its enhancement $\hat{\sigma}_{ij} = f_{ij}(\hat{u}_{(k,l)} + \hat{\varepsilon}_{kl}^P)$, wherein $f_{ij}$ is the constitutive function. For this purpose the element routine $\text{umat}$ must simply call the material routine $\text{umat}$.

$^6$Too many enhanced strains $\hat{\varepsilon}_\beta$ may cause singularity of $\int M_{ij\alpha}E_{ijkl}M_{kl\beta}dV$ in (9).

$^7$The predictor of stress is found from $\sigma_{ij}^P = \sigma_{ij}^e + \hat{\sigma}_{ij}$.

$^8$Contrarily to (6), which is satisfied due to the adjustment of $\hat{\varepsilon}_{kl}$.

$^9$Note that $\hat{\varepsilon}$ may be omitted in differentiations, e.g. $\partial\varepsilon_{ij}/\partial u_{kl} = \partial\varepsilon_{ij}/\partial u_{kl}$. Moreover we will need $\partial \varepsilon_{am}/\partial \varepsilon_{nl} = \lambda_{am} = \partial \varepsilon_{am}/\partial \varepsilon_{nl}$ obtained from $\varepsilon_{am} = \varepsilon_{am}^\varepsilon + \hat{\varepsilon}_{am}$, wherein $\varepsilon_{am}^\varepsilon = \hat{u}_{(k,j)}$. 

November 2015
3.3. Algorithm for ABAQUS® UEL

ABAQUS® calls the element subroutine UEL within the equilibrium iteration. Given the state SVARS at the beginning of the increment and the displacement increment \( \hat{u}_{ij} = DU \) the routine UEL should calculate the out-of-balance forces \( -r_{oi} = \text{RHS} \) at the end of the increment (including the stress increment from the current \( DU \)), see (12) or (13).

Moreover UEL should return the element stiffness \( \left[ K_{\text{eqy}} - L_{\omega i a} D_{a\beta}^{-1} L_{a\gamma} \right] = \text{AMATRX} \), see (13). The contributions from individual elements are aggregated to the global equilibrium equation and ABAQUS® calculates the correction \( c_{ij} \) to the nodal displacements using the element contributions \( \text{AMATRX} \) and \( \text{RHS} \)

\[
\left[ \sum_{\text{elem}} \text{AMATRX}_{\omega ij} \right] c_{ij} = \sum_{\text{elem}} \text{RHS}_{oi} \tag{14}
\]

In order to obtain \( \text{AMATRX} \) and \( \text{RHS} \) and to update stresses (stored in SVARS) the routine UEL proceeds as follows

1. Calculate \( N_\alpha, N_\alpha, i, M_{ij} \) and given \( \hat{u}_{ij} \) find the strain increments \( \hat{\epsilon}_{ij} = \hat{u}_{(i, j)} - N_{ij} \hat{u}_{ij} \) for all GP. For future integrations we will also need the Jacobians \( J^G \) and \( J^L \).

2. Find iteratively \( \hat{\epsilon}_{ij} \) starting from \( \hat{\epsilon}_{ij} = 0 \\
(a) \text{ invoke } \text{umat} \text{ with } \hat{\epsilon}_{ij} = \hat{\epsilon}_{ij}^{\text{u}} + \hat{\epsilon}_{ij}^{\text{e}} \text{ to obtain } \hat{\sigma}_{ij} \text{ and the tangential stiffness } E_{ijkl}. \\
(b) \text{ Find error } \hat{r}_{\alpha} = \int M_{i,j} \hat{\sigma}_{ij} dV \text{ and } D_{\alpha\beta} = \int M_{i,j} \alpha E_{ijkl} M_{k,l} dV. \\
(c) \text{ if } \hat{r}_{\alpha} \hat{r}_{\alpha} < \text{tolerance, then exit the iteration going to 3.} \\
(d) \text{ otherwise calculate the correction } c_{ij} = -D_{\alpha\beta} \hat{r}_{\beta}, \text{ refine } \hat{\epsilon}_{ij}^{\text{e}} = M_{ij} \alpha c_{ij} \text{ and start a new iteration cycle from (a)}

3. Given exact \( E_{ijkl} \), exact \( \hat{\epsilon}_{ij} \), exact \( \hat{\sigma}_{ij} \) and exact \( D_{\alpha\beta} \) for the prescribed \( \hat{u}_{ij} \) find

\[
K_{\omega ij} = \int_{V_T} N_{ij} E_{ijkl} N_{kl} dV \quad \text{and} \quad L_{\omega ij} = \int_{V_T} M_{ij} \alpha E_{ijkl} N_{kl} dV
\]

and using them find the element stiffness and

\[
\text{AMATRX} = \bar{K}_{\omega ij} = K_{\omega ij} - L_{\omega i a} D_{\alpha\beta}^{-1} L_{\gamma j k} \tag{15}
\]
and the out-of-balance forces
\[
RHS = -r_{0y} = \int_V N_{aw} f_y dV + \int_S N_{aw} f_S - \int_V N_{aw,j} (\sigma_{ij}^p + \hat{\sigma}_{ij}^e) dV
\]  
(16)

Note that \( \sigma_{ij}^p = \sigma_{ij}^n + \hat{\sigma}_{ij}^e \) may be modified only in the global equilibrium iteration, i.e. by a a new \( \hat{u}_{yk} \).

4. The current EI may be the final one and hence the master stress should be recovered
\[
\sigma_{ij} = \sigma_{ij}^n + P_{ija} \hat{\sigma}_a \quad \text{with} \quad \hat{\sigma}_a = H^{-1}_{\beta a} R_{\alpha},
\]  
(17)

4. Example

A comparison of EAS elements (CPE4-EAS) to elements with selectively reduced integration (CPE4), with incompatible modes (CPE4I) and with reduced integration (CPE4R) is carried out using the prescribed accumulated strain field \( \varepsilon_{ij}^{acc} \)

\[
\varepsilon_{11}^{acc} = 0.01 x_3^2 / 4, \quad \varepsilon_{22}^{acc} = 0.01 x_1^2 / 4, \quad \gamma_{12}^{acc} = 0.01 x_1 x_2
\]  
(18)

defined upon a freely supported quadratic area \( x_1 \in [0,10] \) and \( x_2 \in [0,10] \) presented in Fig. 1. In this example all strain components vary quadratically in space as expected in HCA applications. The prescribed 2D strain field \( \varepsilon_{ij}^{acc} \) is kinematically compatible, and hence no self stress appears in the analytical solution. The exact displacement field (without rotation) can be easily integrated assuming \( u_{1,2} = u_{2,1} \) and \( u_1(x_1 = 0, x_2 = 0) = 0 \)

\[
u_1^{acc} = 0.01(x_1 x_2^2) / 4 \quad \text{and} \quad u_2^{acc} = 0.01(x_1^2 x_2) / 4
\]  
(19)

For some tests we will rotate this field by \(-45^\circ\).

Figure 1. Statically determinate system and the displacements due to the prescribed accumulated strain field.

A simplified version of the HCA constitutive model with a constant isotropic elastic stiffness (with \( E = 10000 \) and \( \nu = 0 \)) has been used. The results are presented in Tab. 1 and Fig. 3.

CPE4 elements show stress field patterns typical for locking phenomena, Fig. 3. Large self stresses are observed within single elements. Also CPE4I elements are not free from self stresses, specially in the shear component \( \sigma_{12} \). EAS elements show practically
Table 1. Components of self-stress and displacements at \((x_1 = 10, x_2 = 10)\) invoked by the prescribed strain field \(\varepsilon_{ij}^{ext}\) (18). The comparison of different elements and meshes. Large self-stresses \(\sigma_{ij}^{max}\) indicate an error. They are absent in the analytical solution. The hourglass stiffness \(E_{\text{glass}}\) is necessary for reduced integration only.

<table>
<thead>
<tr>
<th>Element</th>
<th>nelem</th>
<th>Rot.</th>
<th>Mesh</th>
<th>(u_{x}^{acc})</th>
<th>(u_{y}^{acc})</th>
<th>(\sigma_{11}^{max})</th>
<th>(\sigma_{22}^{max})</th>
<th>(\sigma_{12}^{max})</th>
<th>(E_{\text{glass}})</th>
</tr>
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<td>CPE4</td>
<td>4</td>
<td>0</td>
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<td>2.435</td>
<td>2.435</td>
<td>859.4</td>
<td>859</td>
<td>1094</td>
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<tr>
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<td>0</td>
<td>irreg.</td>
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<td>2.324</td>
<td>828.7</td>
<td>828.7</td>
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<td></td>
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<tr>
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<td>0</td>
<td>reg.</td>
<td>2.342</td>
<td>2.342</td>
<td>3.109</td>
<td>3.109</td>
<td>1.55</td>
<td>100</td>
</tr>
<tr>
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<td>irreg.</td>
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<td>2.165</td>
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<td>2.396</td>
<td>2.396</td>
<td>8E-13</td>
<td>2.9E-13</td>
<td>312.5</td>
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</tr>
<tr>
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<td>irreg.</td>
<td>2.306</td>
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<td>165.4</td>
<td>165.4</td>
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<tr>
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<td>reg.</td>
<td>3.388</td>
<td>0</td>
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<td>312.5</td>
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<td>reg.</td>
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<td>7E-10</td>
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</tr>
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</table>

no self stresses. The performance of enhanced elements (CPE4-EAS and CPE4I) can be strongly deteriorated in irregular (Fig. 2) meshes, see Tab. 1.

Figure 2. Regular (left), irregular (center), and rotated by \(-45°\) (right) meshes corresponding to \(\text{nelem} = 4\) in Tab. 1.

5. Acknowledgements

The authors are grateful to the DFG (DFG-Forschergruppe FOR 1136) for financial support. Mrs Iryna Loges helped a lot in programming of early MATHEMATICA scripts. The authors are obliged to the colleagues from IBF KIT for fruitful discussions.

References

Figure 3. Numerical self-stress due to locking of a 100-element mesh subject to the initial strain field $\varepsilon^{acc}_{ij}$ prescribed in (18). The chess board patterns are characteristic for parasitic stresses.

Left column: elements with selectively reduced integration (CPE4) generate huge stress.

Middle column: incompatible elements (CPE4I) from ABAQUS® generate considerable shear stress.

Right column: enhanced assumed strain (CPE4-EAS) elements generate almost no parasitic stresses.

Note the differences in the color scales.


