ACCUMULATION OF STRAIN IN SAND DUE TO CYCLIC LOADING UNDER DRAINED TRIAXIAL CONDITIONS

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ABSTRACT

This paper presents the experimental results obtained from high quality drained cyclic triaxial tests. These tests were performed in the framework of a research project on long term settlements of structures due to traffic loads. In order to develop an explicit (N-type) constitutive model for sand under repeated loading numerous drained cyclic triaxial tests have been performed varying average stress, stress amplitude and initial density. These parameters significantly influence the residual strain or stress. Other factors, in particular the shape and the polarization of the strain amplitude, were also investigated and will be discussed in a separate paper in future. The experimentally obtained cyclic flow rule was found to be almost identical with the flow rule from monotonous models (Cam clay and hypoplasticity). The experimental results are summarized by simple empirical formulas which can be implemented into a constitutive model and then to an FE program. Some similar experimental results from the literature are also discussed. Based on the gathered experimental data several existing explicit models can be shown to be oversimplified. A common problem of explicit models related to the initial number of cycles is pointed out.

Key words: Cyclic triaxial tests, accumulation, residual strain, explicit model, sand (IGC: D9)

INTRODUCTION

Cyclic loading leads to an accumulation of strain or stress in soils because the generated stress or strain loops are not perfectly closed. Such cumulative phenomena are of importance in many practical cases. Especially for loose fresh deposits and for large strain amplitudes (i.e. $\gamma_{\text{amplitude}} > 0.001$) a significant accumulation manifests itself in the form of excessive pore water pressure generation (stability) and large settlements (serviceability). In the seismic areas load cycles may lead to accumulation of high pore water pressures and consequently to a dramatic loss of the bearing capacity (liquefaction). Also offshore foundations loaded by storm (wind and/or water) waves may lose their bearing capacity in a similar way. If the intensity of loading is not too high and pore pressures can be dissipated the soil strength is improved and the settlements become the main concern (long-term serviceability). Even if strain amplitude is relatively small a large number of cycles may cause excessive settlements, for instance due to traffic loads (e.g. railways) or vibrating construction site machines (e.g. pile installation). Such cumulative displacement is the main subject of the current study. Of course, some structures are less resistant to settlements than others.

For instance, the settlements of shallow foundations of high speed magnetic levitation train must be kept within an extremely small range in order to ensure the operational requirements. In this case an accurate prediction is required for several decades of intensive traffic.

For the FE predictions of residual settlements a special constitutive model is needed. Here we present some experimental results upon which a so-called explicit model will be based. In explicit (N-dependent) models the residual deformation or accumulation due to a bunch of cycles is calculated by direct formulas within a single increment. The alternative implicit (incremental) models need hundreds of load increments per cycle and the residual settlement appears as a by-product of classically calculated strain - stress loops. The accumulation results from the fact that the cycles are not perfectly closed. The implicit approach is suitable for a relatively low number of cycles (say $N < 50$) and general purpose constitutive models can be used. They must be able to simulate small-strain nonlinearity and subtle structural effects that govern the cyclic behaviour. For a high number (e.g. several thousands) of cycles explicit models turn out to be more useful and accurate because the accumulation of small systematic errors in implicit models becomes comparable to the physical accumulation of interest.

Numerous explicit constitutive models exist in the lit-
eralization techniques are rarely taken into account. The cyclic triaxial tests presented here are used to formulate an explicit model for sand described in detail (inclusive its implementation in an FE program) in Niemunis et al. (2003a). The triaxial tests are more reliable than the conventional ones, explicit models are of
ten based on insufficient experimental data. For these reasons some models assume purely volumetric accumulation, others neglect the influence of the average stress $\sigma_{av}$ from which the amplitude is measured (should not be mixed up with the mean stress $p$). The shape (the openness) and the polarization of the strain amplitude are rarely taken into account.

The existing models in the literature are strongly simplified. They often lack in generality, disregard aspects of invariant notation etc. Usually their authors have a very specific problem and a very special kind of loading in mind. Since the cyclic tests are much more laborious than the conventional ones, explicit models are often based on insufficient experimental data. For these reasons some models assume purely volumetric accumulation, others neglect the influence of the average stress $\sigma_{av}$ from which the amplitude is measured (should not be mixed up with the mean stress $p$). The shape (the openness) and the polarization of the strain amplitude are rarely taken into account.

The cyclic triaxial tests presented here are used to formulate an explicit model for sand described in detail (inclusive its implementation in an FE program) in Niemunis et al. (2003a). The triaxial tests are more reliable and offer more information than the direct simple shear (DSS) ones. For the first, the deformation field in the direct shear apparatus is strongly inhomogeneous (Budhu (1984), Budhu and Britto (1987)). For the second, the accumulation of horizontal stress (often not measured) that goes along with the densification of the DSS sample can significantly falsify the results. For the third the direction (strain ratio) of the accumulated strain cannot be measured in DSS.

Some complementary experimental results from the multiaxial direct simple shear device are presented in Wichmann et al. (2003). In this novel apparatus such effects as the polarization of the strain loop and its openness have been quantified.

NOTATION, PRELIMINARIES

In this paper we use the effective Cauchy stress $\sigma$ (compression positive) and small strain components $\varepsilon_{ij} = -(u_{ij} + u_{ji})/2$. In the triaxial case it is convenient to use the Roscoe variables $p = (\sigma_1 + \sigma_3)/3$ and $q = \sigma_1 - \sigma_3$ and the conjugated strain rates $\dot{\varepsilon}_v = \dot{\varepsilon}_1 + 2\dot{\varepsilon}_3$ and $\dot{\varepsilon}_q = 2/3 (\dot{\varepsilon}_1 - \dot{\varepsilon}_3)$ with the scalar product $p \dot{\varepsilon}_v + q \dot{\varepsilon}_q = \sigma : \dot{\varepsilon}$. We denote the axial component with the index $\Xi_1$ and the lateral components with $\Xi_2$ and $\Xi_3$. Using the word "rate" we mean the derivative with respect to the number of cycles $N$, that is $\dot{\Xi} = \partial \Xi / \partial N$. Stress obliquity may be expressed as $\eta = q/p$ or using the yield function $Y = -I_1 I_2 / I_3$ of Matsuoka and Nakai (1982) as

$$Y = Y_c - 9 \frac{Y_c - 9}{27(3 + \eta)} = \frac{27(3 + \eta)}{(3 + 2\eta)(3 - \eta)}$$

and $Y_c = (9 - \sin^2 \varphi)/(1 - \sin^2 \varphi)$. The inclinations of the critical state line are

$$M_c = \frac{6 \sin \varphi}{3 - \sin \varphi}, \quad M_w = \frac{6 \sin \varphi}{3 + \sin \varphi}$$

for compression and extension and correspond to $Y = 1$. The peak friction angle is denoted by $\varphi_p$ and the critical one as $\varphi$.

For an arbitrary (also tensorial) state variable $\Xi$ we define its average value $\Xi_{av}$ upon a cycle in such way that $\Xi_{av}$ is the center of the smallest sphere that encompasses all states $\Xi_{i}^{(j)}$ of a given cycle. The average value should not be mixed up with the mean value $\frac{1}{N} \sum_{i=1}^{N} \Xi_{i}^{(j)}$. The amplitude is defined as the following operator $\Xi_{ampl}^{(i)} = \max || \Xi_{i}^{(j)} - \Xi_{av} ||$. A more elaborated definition of amplitude (including polarization and openness of the strain loop) is proposed in Niemunis (2003) and Wichmann et al. (2003). At a constant average stress level $\sigma_{av}$ the vertical stress component $\sigma_1$ was cyclically varied with an amplitude $\sigma_{1,ampl}$. It is convenient to use the normalized size of stress amplitude (see Fig. 1)

$$\zeta = \frac{\sigma_{1,ampl}}{\sigma_{av}}.$$  

Fig. 1: Definition of state of stress in a cyclic triaxial test

The strains under cyclic loading can be decomposed into the residual and the resilient portion denoted by the superscripts $\Xi_{acc}$ and $\Xi_{ampl}$, respectively. We abbreviate

$$\varepsilon_{acc} = \varepsilon_{acc}^\Xi \parallel \omega = \varepsilon_{acc}^\Xi / \varepsilon_{acc}.$$  


with $\varepsilon_{\text{acc}}^v$ and $\varepsilon_{\text{acc}}^d$ being the cumulative volumetric and deviatoric strain components, respectively. For triaxial tests the accumulated strain is $\varepsilon_{\text{acc}} = \sqrt{(\varepsilon_{\text{acc}}^v)^2 + (\varepsilon_{\text{acc}}^d)^2}$. The basic variable that dictates the rate of accumulation is chosen to be the strain amplitude. Its definition is based on the deviatoric part of strain only, i.e. instead of $\varepsilon^{\text{ampl}} = (\varepsilon)^{\text{ampl}}$ we prefer

$$\gamma^{\text{ampl}} = (\varepsilon_1 - \varepsilon_3)^{\text{ampl}} = \sqrt[3]{\frac{3}{2}} (\varepsilon^*)^{\text{ampl}}$$

wherein $\varepsilon^*$ denotes the deviatoric part of $\varepsilon$. In our presentation we also use the conventional density index $I_D = (e_{\text{max}} - e)/(e_{\text{max}} - e_{\text{min}})$.

**LITERATURE REVIEW**

There is a general consent among the researchers that with increasing number of cycles $N$ the residual strain $\varepsilon_{\text{acc}}$ increases and the rate of accumulation $\varepsilon_{\text{acc}}^\text{res}$ decreases. Different experimental curves $\varepsilon_{\text{acc}}(N)$, however, are reported. Lentz and Baladi (1980) observed the increase of residual vertical strain $\varepsilon_{\text{acc}}^v$ to be proportional to the logarithm of the number of cycles $N$ in cyclic triaxial tests. Suiker (1999) stated from cyclic triaxial tests on gravel and sand that the accumulation rate $\varepsilon_{\text{acc}}$ decreases proportional to $1/N$ with two proportionality coefficients $c_1$ for $N < 1000$ and $c_2$ for $N > 1000$ with $c_1 > c_2$. Helin et al. (2000) observed similar proportionality coefficients but with $c_2 > c_1$ obtained from cyclic triaxial tests on medium and fine sand, i.e. the accumulated strain increases faster than the logarithm of the number of cycles. Similar conclusions were drawn by Gotschol (2002) from cyclic triaxial tests on gravel.

The effect of strain amplitude was mainly studied in cyclic simple shear tests. Yould (1972) reported a considerable increase of the rate of accumulation with increasing shear strain amplitude $\gamma^{\text{ampl}}$. At amplitudes lower than $\gamma^{\text{ampl}} = 10^{-4}$ he observed no accumulation of residual strain. Similar experimental results were obtained by Silver and Seed (1971). They claimed that the densification process under cyclic loading can be easier described using the shear strain amplitude $\gamma^{\text{ampl}}$ instead of the amplitude of shear stress $\tau^{\text{ampl}}$. Sawicki and Świdziński (1987, 1989) performed cyclic simple shear tests with different shear strain amplitudes. For a sand at a given initial density they demonstrated a unique curve $\varepsilon_{\text{acc}}^v(N)$ with $N = N (\gamma^{\text{ampl}}/2)^2$. In other words independently of the applied amplitude the curves $\varepsilon_{\text{acc}}^v(N)$ fall together to a so-called "common compaction curve". Marr and Christian (1981) studied the influence of the mean stress, the stress amplitude and the initial density on the cyclic triaxial behavior of sand. From their results one could derive a relationship $\varepsilon_{\text{acc}} = a \zeta^b$ with the exponent $b$ in the range $1.91 \leq b \leq 2.32$ for $N$ between 10 and 1,000.

In the above mentioned studies only deviatoric strain amplitude was chosen as an independent parameter. On the contrary Ko and Scott (1967) performed tests on cubical specimens with an isotropic stress amplitude in order to evaluate its effect on the accumulation. After a small initial compaction no further accumulation was observed and so the volumetric amplitude can be neglected.

Several experimental studies with cyclic simple shear tests (e.g. Youd (1972), Silver and Seed (1971)) ended with the conclusion that the average mean pressure $p^{\text{av}}$ does not influence the accumulation of residual strain. Marr and Christian (1981) performed a series of cyclic triaxial tests with identical values of the average stress ratio $(q^{\text{av}} = q^{\text{av}}/p^{\text{av}} = 0.43)$, the amplitude ratio $(\zeta = 0.19)$ and the initial density ($D_r = 28\%$) but with different average mean pressures (140 kPa $\leq p^{\text{av}} \leq 420$ kPa). For a higher $p^{\text{av}}$ slightly higher accumulated strains were observed. However, different shear strain amplitudes in the tests due to the stress-dependence of stiffness were not taken into account. From tests with nearly identical average mean pressures (217 kPa $\leq p^{\text{av}} \leq 250$ kPa), amplitude ratios ($0.18 \leq \zeta \leq 0.21$) and initial densities ($D_r = 28\%$) it became obvious, that with increasing $q^{\text{av}}$ ($0 \leq q^{\text{av}} \leq 0.92$ were tested) the accumulation increases and becomes more deviatoric.

From numerous cyclic simple shear tests (e.g. Youd (1972), Silver and Seed (1971)) and cyclic triaxial tests (e.g. Marr and Christian (1981)) it was evident that initial soil density is an important parameter for the prediction of accumulation. Loose samples showed higher accumulation rates.

The reports on the influence of the frequency of the cyclic loading are somewhat contradictory. Youd (1972) has not observed any influence in cyclic simple shear tests between 0.2 and 1.9 Hz. Similarly, no influence of the frequency was detected by Shenton (1985) in cyclic triaxial tests with frequencies between 0.1 Hz and 30 Hz. Cyclic triaxial tests on gravel performed by Kempfert et al. (2000) showed, however, that with increasing frequency the permanent strain due to the first load cycle increased and the accumulation rate during the following cycles decreased.

The direction of accumulation was studied by Luong (1982) in drained cyclic triaxial tests. Small packages with 20 load cycles were applied successively to the same sand specimen. The average stress was varied from package to package and the accumulated strain was monitored. The existence of a so called "characteristic threshold (CT)" line in the $p$-$q$-plane was postulated such that $\sigma^{\text{av}}$ below the CT line lead to densification and $\sigma^{\text{av}}$ above the CT line to dilative accumulation. Chang and Whitman (1988) found that the direction of accumulation could be well expressed using the flow rule of the modified Cam Clay model $\omega = (M_r^2 - (p^{\text{av}})^2)/(2p^{\text{av}})$. They proposed to identify Luong’s CT line with the critical state line. The CT line was shown to be independent of the density. Moreover Chang and Whitman demonstrated that the ratio of volumetric and deviatoric residual strain does not depend
either on \( p^v \) or \( \zeta \). However, the specimens tested by Chang and Whitman were subject to \( N = 1,050 \) cycles only. A slight increase of \( \omega \) with \( N \) could be concluded from the diagrams presented by Chang and Whitman.

**TESTING DEVICES, PREPARATION OF SAMPLES AND TESTED MATERIALS**

![Cyclic Triaxial Test Device Diagram](image)

**Fig. 2: Scheme of cyclic triaxial test device**

Two cyclic triaxial devices one for wet and one for dry samples were used. A scheme of the first one is presented in Fig. 2. The top plate of the pressure cell consisted of two parts, the inner and the outer one. This allowed specimen preparation (sealing, measuring geometry) before the plexiglas cylinder and the outer part were mounted and screwed together with the base plate. The load piston was guided in a ball bearing fixed to the inner part of the top plate. In order to minimize friction a special sealing was used at the entrance of the load piston into the pressure cell. The upper end plate was rigidly connected to the load piston in order to prevent tilting. The actual vertical force was measured by a force transducer inside the pressure cell below the upper specimen end plate. A displacement transducer of a high accuracy was used to monitor the vertical deformation. The volume changes were measured by the squeezed out pore water. A differential pressure transducer compared the water level in the measuring column (connected with the specimen drainage) with a reference pressure generated by the constant water level in another column. Cell pressure and back pressure were independently controlled by means of pressure transducers.

The second triaxial device was used for dry specimens. In this cell lateral deformations were measured by three pairs of local non-contact displacement transducers (LDT) mounted evenly over the height of the specimen. The vertical load was measured with a force transducer above the upper specimen end plate. A detailed description of this triaxial cell was given by Brigioni et al. (1996).

In both triaxial devices the average and the cyclic vertical force were controlled by a pneumatic loading system. The desired values of the vertical force (offset, amplitude and frequency) were given by a function generator. A controller compared the desired and the actual forces and adjusted a valve regulating the air pressure applied to the pneumatic cylinder placed on the top of the loading frame. The pneumatic cylinder transmitted the force to the piston.

In all tests the specimens were subject to 100,000 cycles at a frequency of 1 Hz. The signals of all transducers were sampled over the period of five complete cycles after a idle phase of \( \Delta N \) cycles. The distance \( \Delta N \) between the readings increased with \( N \). From the recorded signals all components of the residual strain and the strain amplitude could be calculated.

In order to prepare a specimen a thin rubber membrane was mounted at the lower specimen end plate and sealed with rubber rings. A mould consisting of two half-cylinders was screwed together and the membrane was sucked towards these mould by vacuum. The sand was dry pluviated from a funnel into the mould. The funnel was lifted so that the fall height remained constant. Different initial densities were achieved by varying the outlet diameter of the funnel and the fall height. The inner part of the top plate of the pressure cell with the load piston was mounted. The upper end plate was let down onto the sand surface and the membrane was rolled up and sealed. Next a vacuum of 50 kPa was applied to the grain skeleton in order to stabilize the sand specimen. The half-cylinders of the mould could then be removed. The initial density was calculated from the measured volume and the mass of the specimen. Finally the plexiglas cylinder was mounted and fixed with the outer part of the top plate of the pressure cell. The cell was filled with water leaving a small air cushion below the top plate. The vacuum in the sample was gradually replaced by the cell pressure so that the effective stress was held constant. The cell pressure was pneumatically controlled via the air cushion.

If the specimen was supposed to be water-saturated it was flushed with carbon dioxide in order to replace the air and after that saturated with de-aired water. The drainage of the specimen was connected to the water column of the volume measuring unit. The cell pressure and the back pressure (applied on the grain skeleton via the volume measurement unit) were raised simultaneously until a back pressure was reached at which all remaining bubbles were solved in water. The cell pressure was kept slightly higher than the back pressure. In most tests a back pressure of 300 kPa was used. In tests
with high effective lateral pressures the back pressure was 200 kPa, since the pressure cell could sustain only 500 kPa. In order to optimize full saturation specimens stand under this low effective stress over night until the next day.

The quality of saturation (B > 0.95) was checked by the well known B-value of Skempton, cf. e.g. Lambe (1969). After that the cell pressure was increased and then the vertical stress was applied to the desired value $\sigma^{av}$ while the deformations were continuously measured. The subsequent cyclic variation of the vertical stress was preceded by a consolidation period of one hour.

All tests were performed on the same medium sand. The grain distribution curve, the maximum and minimum void ratios and the critical friction angle are presented (1969). After that the cell pressure was increased and then the vertical stress was applied to the desired value $\sigma^{av}$, while the deformations were continuously measured. The quality of saturation (B > 0.95) was checked by the well known B-value of Skempton, cf. e.g. Lambe (1969). After that the cell pressure was increased and then the vertical stress was applied to the desired value $\sigma^{av}$ while the deformations were continuously measured. The subsequent cyclic variation of the vertical stress was preceded by a consolidation period of one hour.

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The detailed forms of the functions are given in Table 1. The main objective of this paper was to separate the effects of the above mentioned six parameters, to provide experimental evidence for the assumed dependencies and to determine the material constants.

The total accumulated strain $\varepsilon^{acc}$ which is used throughout this paper is obtained from integration over the number of cycles, i.e. $\varepsilon^{acc} = \int \varepsilon^{acc}dN$. In our explicit model (Niemunis et al. (2003a)) the rate $\dot{\varepsilon}^{acc}$ is the basic variable.

The irreversible deformation generated by the first so-called irregular cycle that is applied immediately after monotonic loading must be treated separately. This accumulation is usually much larger than the accumulation caused by the regular (second and subsequent) cycles (Fig. 4), at least for fresh pluviated samples. For the sake of simplicity the accumulation from the irregular cycle is disregarded in the explicit description. The cycle $N = 1$ refers therefore to the first regular cycle. The explicit FE calculation (Niemunis et al. (2003a)) of accumulation is commenced at $N = 2$ or later. The irregular cycle and the first regular cycle are calculated implicitly. This is necessary because one needs a representative information about one full regular cycle to estimate the strain amplitude, its polarization etc. The irregular cycle is not suitable for this purpose. Note that if the irregular cycle happened to be similar to the subsequent ones (to be actually regular) the above mentioned precaution in the FE calculation is redundant but safe.

![Fig. 3: Grain distribution curve, maximum and minimum void ratios and critical friction angle of the used medium sand](image)

**PROPOSED EXPLICIT MATERIAL MODEL**

The general stress-strain relation has the form

$$\sigma = E : (\varepsilon - \varepsilon^{acc})$$

wherein $E$ denotes the elastic stiffness calculated for the given stress $\sigma$. The rate of strain accumulation $\dot{\varepsilon}^{acc}$ is proposed to be

$$\dot{\varepsilon}^{acc} = \dot{\varepsilon}^{acc} \mathbf{m} = f_{ampl} \dot{f}_N f_p f_Y f_e f_\pi \mathbf{m}$$

with the direction expressed by the unit tensor $\mathbf{m}$ and with the intensity $\dot{\varepsilon}^{acc} = ||\dot{\varepsilon}^{acc}||$ given by six partial functions $f$. For triaxial tests $\omega = \sqrt{\frac{3}{2}} \text{tr} \mathbf{m}/||\mathbf{m}^||$ holds.

The intensity of strain accumulation $\dot{\varepsilon}^{acc}$ depends on the strain amplitude $\varepsilon^{ampl}$, the number of cycles $N$, the average stress $p^{av}$, $Y^{av}$, the void ratio $e$, the cyclic strain history $\pi$ and the shape of the strain loop. In the proposed expression (7) six partial functions describe the following contributions

- $f_{ampl}$: strain amplitude $\varepsilon^{ampl}$
- $f_N$: number of cycles $N$
- $f_p$: average mean stress $p^{av}$
- $f_Y$: average stress ratio $Y^{av}$
- $f_e$: void ratio $e$
- $f_\pi$: polarization of strain amplitude

In the following section numerous results of cyclic triaxial tests are presented.
INFLUENCE OF THE SHEAR STRAIN AMPLITUDE

In general multiaxial case the shape of the strain loop can be described with a special definition of the strain amplitude. This definition is presented by Niemunis (2003) and Niemunis et al. (2003b). For the triaxial tests the amplitude is uniaxial and the general definition is identical with our \((\varepsilon)^{\text{ampl}}\).

The shear strain amplitude \(\gamma^{\text{ampl}}\) was studied in a series of tests with varying stress amplitude between \(0.06 \leq \zeta \leq 0.47\) (12 kPa \(\leq \sigma_1^{\text{ampl}} \leq 94\) kPa). The average stress was \(p^{\text{av}} = 200\) kPa, \(\eta^{\text{av}} = 0.75\) in all tests and the initial relative density was \(0.55 \leq I_D0 \leq 0.64\), wherein "initial" means after the irregular cycle. Specimens were tested in the dry condition.

![Diagram](image.png)

Fig. 5: Accumulated strain \(\varepsilon^{\text{acc}}\) as a function of the number of cycles \(N\) in tests with different shear strain amplitudes \(\gamma^{\text{ampl}}\) (all tests: \(p^{\text{av}} = 200\) kPa, \(\eta^{\text{av}} = 0.75\), \(0.55 \leq I_D0 \leq 0.64\))

Keeping \(\zeta(N)\) constant a small decrease of shear strain amplitude \(\gamma^{\text{ampl}}\) with \(N\) was noticed, especially during the first 100 cycles. This variation is disregarded and mean values are used in the following evaluations. A linear proportionality between stress amplitude ratio \(\zeta\) and shear strain amplitude \(\gamma^{\text{ampl}}\) was observed. Fig. 5 presents the accumulated strain \(\varepsilon^{\text{acc}}\) as a function of the number of cycles at different shear strain amplitudes \(\gamma^{\text{ampl}}\) in a semi-logarithmical diagram. Only regular cycles are shown. The accumulated strain was found to increase almost linearly with the logarithm of the number of cycles up to \(N \approx 1,000\) and then overlinearly. The function \(f_N\) is proposed in a successive section. From Fig. 5 it can be clearly seen that higher shear strain amplitudes \(\gamma^{\text{ampl}}\) cause a larger accumulation \(\varepsilon^{\text{acc}}\). In Fig. 6 the accumulated strain is presented as a function of the square of the shear strain amplitude for different numbers of cycles. Accumulation \(\varepsilon^{\text{acc}}\) turns out to increase proportionally to \((\gamma^{\text{ampl}})^2\), independently of \(N\). The following amplitude function

\[
\sigma_{\text{ampl}} = \left(\frac{\gamma^{\text{ampl}}}{\gamma_{\text{ref}}}\right)^2
\]

is proposed with the reference amplitude \(\gamma^{\text{ref}} = 10^{-4}\). Function (8) is in agreement with the exponents that have been determined from the tests of Marr and Christian (1981).

In Fig. 7 the accumulated strain is shown as a function of \(N\ \left(\gamma^{\text{ampl}}\right)^2\). Evidently the "common compaction curve" proposed by Sawicki and Świdziński (1987, 1989) on the basis of DSS could not be confirmed by our triaxial results.

![Diagram](image.png)

Fig. 6: Accumulated strain \(\varepsilon^{\text{acc}}\) as a function of the square of the shear strain amplitude \((\gamma^{\text{ampl}})^2\) for different numbers of cycles \(N\) (all tests: \(p^{\text{av}} = 200\) kPa, \(\eta = 0.75\), \(0.55 \leq I_D0 \leq 0.64\))

![Diagram](image.png)

Fig. 7: Accumulated strain \(\varepsilon^{\text{acc}}\) in dependence on the weighted number of cycles \(N\ \left(\gamma^{\text{ampl}}\right)^2\) as proposed by Sawicki and Świdziński in tests with different shear strain amplitudes \(\gamma^{\text{ampl}}\) (all tests: \(p^{\text{av}} = 200\) kPa, \(\eta = 0.75\), \(0.55 \leq I_D0 \leq 0.64\))

Fig. 8 presents the strain ratio \(\omega\) as a function of shear strain amplitude. At higher values of \(\gamma^{\text{ampl}}\) the strain ratio is almost independent of \(\gamma^{\text{ampl}}\). At smaller shear strain amplitudes the accumulated volumetric and deviatoric strains were very small and therefore the values of \(\omega\) could be inaccurate. Thus the direction of accumulation is thought to be independent of the shear strain amplitude. This is in accordance with the experimental
results of Chang and Whitman (1988). Plots of $\omega$ over $N$ demonstrate an increase of the volumetric part of the accumulated strain with the number of cycles.

**Fig. 8:** Strain ratio $\omega$ as a function of shear strain amplitude $\gamma_{\text{ampl}}$ for different numbers of cycles $N$ (all tests: $p^\text{av} = 200$ kPa, $\eta = 0.75$, $0.55 \leq I_D0 \leq 0.64$)

**Fig. 9:** Strain amplitudes in tests with different average mean pressures (all tests: $\eta^\text{av} = 0.75$, $0.30 \leq \zeta \leq 0.61$, $0.61 \leq I_D0 \leq 0.69$)

**Fig. 10:** Accumulated strain $\varepsilon^\text{acc}$ divided by the amplitude function $f_{\text{ampl}}$ in dependence on the average mean pressure $p^\text{av}$ for different numbers of cycles (all tests: $\eta^\text{av} = 0.75$, $\zeta = 0.30$, $0.61 \leq I_D0 \leq 0.69$)

**INFLUENCE OF THE AVERAGE STRESS**

The average stress $\sigma^\text{av}$ was varied between 50 kPa $\leq p^\text{av} \leq 300$ kPa and $0.25 \leq \eta^\text{av} \leq 1.375$ keeping $\zeta = 0.30$ constant. The specimens were prepared within a small range of initial densities ($0.57 \leq I_D0 \leq 0.69$) and tested in the saturated condition.

The results of a series of six tests with identical average stress ratio ($\eta^\text{av} = 0.75$) but different average mean pressures keeping $\zeta = 0.30$ constant are presented in Figs. 9 - 12. The strain amplitudes slightly increased with increasing mean pressure due to the well known dependence of stiffness $G \sim p^{0.75}$, i.e. slightly under-proportional to $p$.

The average mean pressure $p^\text{av}$ was kept constant for different numbers of cycles $N$. The function $f_p$ is proposed in the simple form

$$f_p = \exp\left[ -C_p \left( \frac{p^\text{av}}{p_{\text{atm}}} - 1 \right) \right]. \quad (9)$$

The atmospheric pressure $p_{\text{atm}} = 100$ kPa is used as a reference for which $f_p = 1$ holds. Equation (9) was fitted to the data in Fig. 10 for different numbers of cycles. The corresponding curves are shown as solid lines in Fig. 10. Actually the parameter $C_p$ could be made dependent on the number of cycles (Fig. 11). The points in Fig. 11 would be better approximated with $C_p(N)$ of the form

$$C_p(N) = \frac{1 + 0.557 N^{0.244}}{1 + 0.0304 N^{0.244}}, \quad (10)$$

which is shown as solid line in Fig. 11. However in order to keep the number of material constants manageable, a constant value of $C_p = 0.50$ was chosen. The dependence $\varepsilon^\text{acc}(N)$ is in agreement with the shape of the curves shown in Fig. 5.

The direction $m$ or $\omega$ of the accumulation was found to be relatively insensitive to changes in the average mean pressure (Fig. 12). Only a slight increase of strain ratio $\omega$ with decreasing average mean pressure could be observed. Moreover a slight increase of $\omega$ with $N$ could be monitored. Of course, having reached the state of maximum density this tendency is expected to inverse, i.e. only deviatoric accumulation is possible.

The results of a series of eleven tests with $0.375 \leq \eta^\text{av} \leq 1.375$ ($0.088 \leq Y^\text{av} \leq 1.243$) and $p^\text{av} = 200$ kPa $= \text{const}$ and $\zeta = \text{const}$ are presented in Figs. 13 - 15.

The strain amplitudes slightly decrease with increasing $\eta^\text{av}$ as shown in Fig. 13. The accumulated strain normalized by the amplitude function at stress ratio $Y^\text{av}$ for
Fig. 11: Fitting parameter $C_p$ as a function of the number of cycles

Fig. 12: Strain ratio $\omega$ as a function of average mean pressure $p_{av}$ for different numbers of cycles $N$ (all tests: $\eta_{av} = 0.75$, $\zeta = 0.30$, $0.61 \leq I_{D0} \leq 0.69$)

Fig. 13: Strain amplitudes in tests with different average stress ratios $\eta_{av}$ (all tests: $p_{av} = 200$ kPa, $\zeta = 0.30$, $0.57 \leq I_{D0} \leq 0.67$)

Fig. 14: Accumulated strain $\varepsilon_{acc}$ divided by the amplitude function $f_{ampl}$ in dependence on the average stress ratio $Y_{av}$ for different numbers of cycles (all tests: $p_{av} = 200$ kPa, $\zeta = 0.30$, $0.57 \leq I_{D0} \leq 0.67$)

different numbers of cycles is presented in Fig. 14. The accumulation increases with $Y_{av}$. Above $Y_{av} = 1.243$ ($\eta = 1.375$) the rate of accumulation suddenly explodes reaching nearly 20% after 100,000 cycles. This can be explained by the fact that for $Y_{av} = 1.243$ the maximum deviatoric stress $q_{max} = q_{av} + q_{ampl}$ during a load cycle lies above $M_c(\varphi_p)$. The peak friction angle $\varphi_p = 36.8^\circ$ corresponds to $I_D = 0.63$ determined from static triaxial tests. Below this limit the function $f_Y$ can be well approximated by

$$f_Y = \exp \left( C_Y Y_{av} \right)$$

(11)

The material constant $C_Y = 2.05$ can be shown to be approximately independent on the number of cycles. For isotropic stresses ($Y_{av} = 0$) $f_Y = 1$ holds.

Fig. 15: Strain ratio $\omega$ as a function of average stress ratio $\eta_{av}$ for different numbers of cycles $N$ (all tests: $p_{av} = 200$ kPa, $\zeta = 0.30$, $0.57 \leq I_{D0} \leq 0.67$)

Fig. 15 shows the measured values of strain ratio $\omega$ plotted over $\eta_{av}$. A decrease of $\omega$ with $\eta_{av}$ is evident. The critical stress ratio $M_c(\varphi)$ was determined
from static tests. For \( \eta^{\text{av}} < M_c(\phi) \) densification was observed. Specimens with an average stress above the critical state line exhibited a dilative behavior. This is in accordance with the experimental results of Luong (1982) and Chang and Whitman (1988). The dependence \( \omega(\eta^{\text{av}}) \) conforms with the flow rule of the modified Cam Clay model as suggested by Chang and Whitman (1988). The agreement with the hypoplastic theory, Niemunis (2003), Fig. 15, is also satisfactory. Both flow rules (for static loads) describe well the direction of accumulation observed in the cyclic triaxial tests at \( N = 100,000 \).

The direction of accumulation at different average stresses \( \sigma^{\text{av}} \) is also shown in Fig. 16. The direction of accumulation is expressed by a unit vector with starting point at \( (\overline{p}, \overline{\epsilon}) \). Only some of the performed tests are shown.

**INFLUENCE OF THE INITIAL DENSITY**

In the final series of tests different initial void ratios \( (0.580 \leq \epsilon_0 \leq 0.688, 0.63 \leq I_{D0} \leq 0.99) \) were used. The average stress \( (p^{\text{av}} = 200 \text{ kPa}, \eta^{\text{av}} = 0.75) \) and the amplitude ratio \( (\zeta = 0.30) \) were held constant within this series. The specimens were tested under water-saturated conditions.

Fig. 17 presents the strain amplitudes as a function of the initial void ratio \( \epsilon_0 \). The condition \( \zeta = \text{const} \) implies slightly higher strain amplitudes for lower initial densities, because \( G(\epsilon) \) decreases with void ratio \( \epsilon \). Fitting the void ratio function

\[
G(\epsilon) \sim \frac{(a - \epsilon)^2}{1 + \epsilon}
\]

of Hardin and Black (1966) to the curve of shear strain amplitude over void ratio \( \gamma \sim 1/G(\epsilon) \) leads to \( a = 1.59 \). In dynamic tests on the same sand (tests in a resonant column device and comparative tests measuring wave velocities with piezoelectric elements in a triaxial cell, Wichtmann and Triantafyllidis (2003a, 2003b)) the parameter \( a \) of equation (12) was determined as \( a = 1.46 \) for amplitudes \( \gamma^{\text{ampl}} \approx 10^{-6} \). Thus the relationship \( \gamma^{\text{ampl}}(\epsilon) \) found in the cyclic triaxial tests seems realistic.

As it can be seen from Fig. 18 the rate of accumulation is high for loose soils. The accumulated strains in Fig. 18 are normalized with the amplitude function (8) therefore the increase of the accumulation rate with increasing void ratio in Fig. 18 cannot be attributed to the increase of shear strain amplitude with void ratio.
The following function (similar to (12)) was found to describe the cyclic void ratio dependence:

\[ f_e = \frac{(C_e - e)^2}{1 + e} \frac{1 + e_{ref}}{(C_e - e_{ref})^2} \]  

(13)

The material constant \( C_e = 0.52 \) was found to be independent of the number of cycles. The reference void ratio is \( e_{ref} = e_{max} = 0.874 \), i.e. \( f_e = 1 \) holds true for \( e = e_{max} \). The data in Fig. 18 has a larger scatter than in the presentations of the cyclic triaxial tests with variation of stress amplitude and mean state of stress, respectively. We attribute it to the scatter of measured shear strain amplitudes (Fig. 17) rather than to the scatter of the data points \( e^{acc}(e) \). Slightly different outlet diameters of the funnel may have contributed to this scatter.

The direction of accumulation was found to be independent of the void ratio for the range of densities studied (Fig. 19). However for dense specimens (\( I_{D0} > 0.90 \)) discrepancies in the strain ratios \( \omega \) were large. These strain ratios are excluded from Fig. 19. The scatter can be attributed to \( \omega \) calculated as a quotient of two very small strains which is less reliable than the strain ratio in tests with higher accumulation rates.

The curve presented as solid line in Fig. 20 corresponds to equation (14) with \( C_{N1} = 2.0 \cdot 10^{-4} \), \( C_{N2} = 0.549 \) and \( C_{N3} = 5.7 \cdot 10^{-5} \). Equation (14) alone describes \( \varepsilon^{acc}(N) \) for \( \varphi = \varphi_{ref}, p^0 = p_{atm}, Y^w = 0 \) and \( e = e_{max} \), i.e. for all other functions equal 1.

![Accumulated strain up to the cycle](image)

**Fig. 19:** Strain ratio \( \omega \) as a function of actual void ratio \( e \) for different numbers of cycles \( N \) (all tests: \( p^0 = 200 \text{ kPa}, \eta^w = 0.75, \zeta = 0.30 \))

**DIFFICULTIES IN ASSESSMENT OF CYCLIC HISTORY**

Some experimental observations indicate that the influence of cyclic deformation history is quite complex and needs a precise description with a tensorial state variable. The rate of accumulation depends on one hand on the number of preloading cycles and on the other hand on their polarization, i.e. the orientation of the strain loop during the recent several hundreds of cycles. The latter effect is discussed in detail by Wichtmann et al. (2003).

The function \( f_e \) in equation (7) describes the influence of the history of the strain polarization. A novel “back polarization” tensor \( \pi \) as well as the function \( f_e \) are discussed in detail by Wichtmann et al. (2003) and Niemunis et al. (2003a). For uniaxial cyclic loading in a triaxial cell \( f_e = 1 \) can be set.

For the present the cyclic history is lumped together into a single scalar variable \( N_0 \) that enterers the equation for \( f_e \) as the initial number of cycles. This \( N_0 \) turns out to be the crucial state variable in the calculation of cyclic densification. Two samples under the same stress \( \sigma_0 \) and with the same void ratio \( e_0 \) may behave quite differently during cyclic loading. If one of them was pluviated to have at once the desired void ratio \( e_0 \) then we
set \( N_0 = 0 \). The other sample could be loosely pluviated with \( \epsilon > \epsilon_0 \) and cyclically densified to reach \( \epsilon_0 \) after \( N_0 > 0 \) cycles. The rate of densification of the second sample during subsequent cycles is much smaller than that of the first one. This structural effect is quite strong and, unfortunately despite numerous efforts, it could not be correlated with small strain stiffness as initially speculated (Niemunis and Triantafyllidis (2000), Wichtmann and Triantafyllidis (2003a, 2003b)). Other possible correlations (e.g. with damping properties or liquefaction potential) are being investigated. As an alternative to direct measurements for practical purposes the in-situ value of \( N_0 \) could be estimated from the inverse analysis of the rate of densification acquired from surface settlements due to a vibratory machine with known working parameters.

### SUMMARY AND CONCLUSIONS

Numerous cyclic triaxial tests with varying mean stress, amplitude and initial density were performed. Based on the obtained results (and additional tests in the cyclic multiaxial direct simple shear device) components of an explicit material model have been formulated. Details of the constitutive model (e.g. special tensorial formulation of strain amplitude considering multiaxial loading) and the simple shear tests are presented by Wichtmann et al. (2003) and Niemunis et al. (2003a). The main conclusions from these tests are:

- The accumulated strain \( \varepsilon_{\text{acc}} \) increases faster than the logarithm of the number of cycles \( N \).
- The strain accumulation is proportional to the square of the shear strain amplitude \( \varepsilon_{\text{acc}} \sim (\gamma_{\text{ampl}})^2 \).
- The so-called "common compaction curve" (Sawicki and Świdziński (1987, 1989)) could not be reproduced in the triaxial tests.
- The direction of accumulation does not depend on the applied shear strain amplitude.
- The volumetric part of the direction of accumulation slightly increases with the number of cycles.
- The accumulated strain \( \varepsilon_{\text{acc}} \) increases with decreasing average mean pressure \( p^{\text{av}} \) keeping other factors (also \( \epsilon \)) constant.
- The dependence of the direction of accumulation on the average mean pressure \( p^{\text{av}} \) is negligible.
- The accumulated strain \( \varepsilon_{\text{acc}} \) increases with increasing average stress ratio \( \eta^{\text{av}} \).
- If the maximum deviatoric stress during a cycle \( q^{\max} = q^{\text{av}} + q^{\text{ampl}} \) exceeds the static failure line the rate of accumulation rapidly increases.

<table>
<thead>
<tr>
<th>Function</th>
<th>Mat. constants</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_{\text{ampl}} = \left( \frac{\gamma_{\text{ampl}}}{\gamma_{\text{ref}}} \right)^2 )</td>
<td>( \gamma_{\text{ampl}} )</td>
</tr>
<tr>
<td>( f_N = C_{N1} [\ln (1 + C_{N2} N) + C_{N3} N] )</td>
<td>( C_{N1} )</td>
</tr>
<tr>
<td>( f_N = \frac{C_{N1} C_{N2}}{1 + C_{N2} N} + C_{N1} C_{N3} )</td>
<td>( C_{N2} )</td>
</tr>
<tr>
<td>( f_P = \exp \left[ -C_p \left( \frac{p^{\text{av}}}{p_{\text{atm}}} - 1 \right) \right] )</td>
<td>( p_{\text{atm}} )</td>
</tr>
<tr>
<td>( f_Y = \exp (C_Y Y^{\text{av}}) )</td>
<td>( C_Y )</td>
</tr>
<tr>
<td>( f_c = \frac{(C_c - \epsilon)^2}{1 + \epsilon} \frac{1 + \epsilon_{\text{ref}}}{(C_c - \epsilon_{\text{ref}})^2} )</td>
<td>( C_c )</td>
</tr>
<tr>
<td>( \epsilon_{\text{ref}} )</td>
<td>( 0.874 )</td>
</tr>
<tr>
<td>( f_{\pi} = 1 + C_{\pi1} \left( 1 - \left( \frac{\overrightarrow{\mathbf{A}} \cdot \pi}{C_{\pi2}} \right)^{C_{\pi2}} \right) )</td>
<td>( C_{\pi1} )</td>
</tr>
<tr>
<td>( \hat{\pi} = C_{\pi3} \left( \frac{\overrightarrow{\mathbf{A}} \cdot \pi}{C_{\pi3}} \right) )</td>
<td>( C_{\pi2} )</td>
</tr>
<tr>
<td>( \hat{C}_{\pi3} )</td>
<td>( 0.0004 )</td>
</tr>
</tbody>
</table>

Table 1: Summary of the partial functions \( f_i \) and a list of the material constants \( C_i \) for the tested sand.

- The direction of accumulation is in agreement with the flow rule of the modified Cam clay model and the hypoplastic flow rule, at least for medium dense and dense sand.
- The accumulated strain \( \varepsilon_{\text{acc}} \) grows with increasing void ratio.
- The direction of accumulation does not significantly depend on the void ratio.

Table 1 presents the functions \( f_i \) and the material constants \( C_i \) of the proposed model.

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### REFERENCES


