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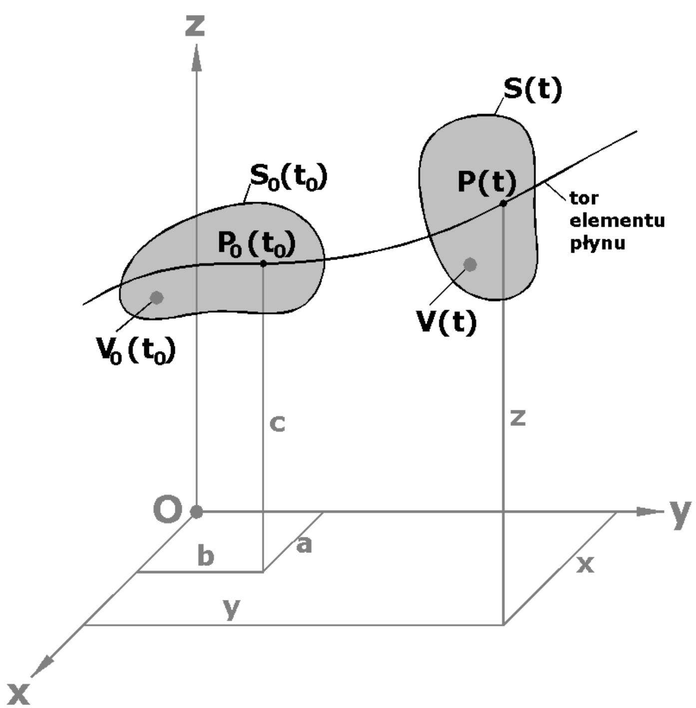
Lecture No. 1: Review of the Principles of Fluid Mechanics

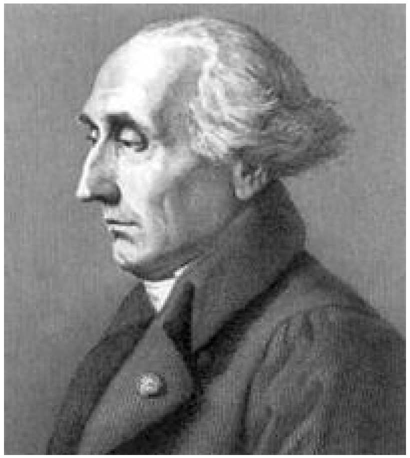
Methods of description of the fluid motion

Lagrange method is based on describing the motion in space of a certain selected mass of fluid, composed always of the same molecules.



V – volume of a certain mass of fluid (fluid volume) surrounded by the surface *S*, **impenetrable** for the fluid elements

Fluid mass moves from location V_0 at time t_0 to location *V* at time *t*.



Joseph Lagrange
1736 - 1813

Fluid element *P* constituting a part of the volume *V* moves in space, tracing the trajectory of the element, which may be described by the following equations with time *t* as parameter:

$$x = x(a, b, c, t)$$

$$y = y(a, b, c, t)$$

$$z = z(a, b, c, t)$$

By changing the quantities *a*, *b* i *c* in the equations different fluid elements may be described

The quantities describing fluid motion depend on *a*, *b*, *c*, *t* in the same way:

$$\bar{u} = \bar{u}(a, b, c, t)$$

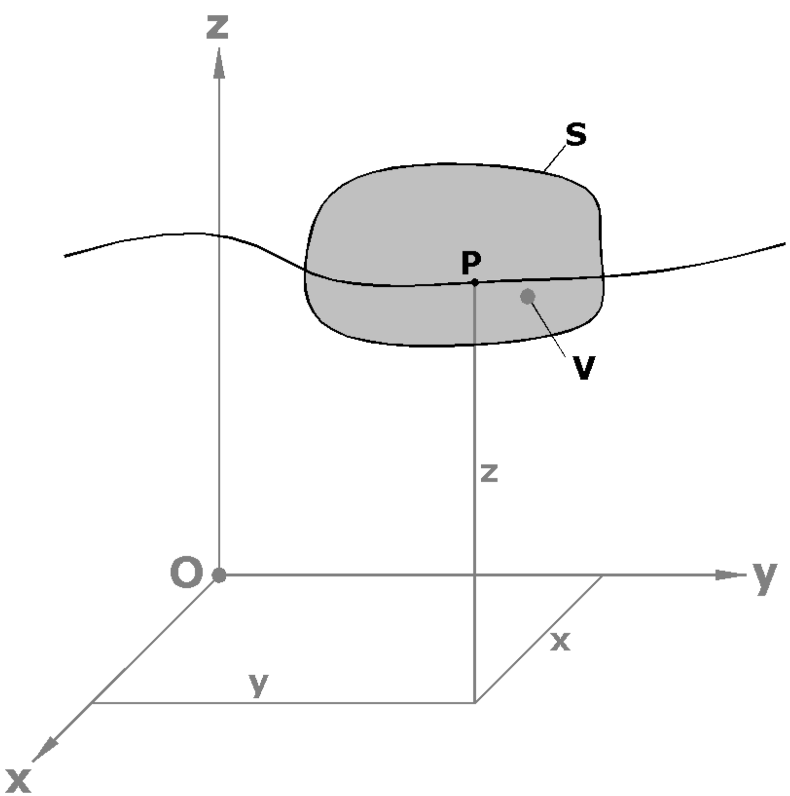
$$p = p(a, b, c, t)$$

$$\rho = \rho(a, b, c, t)$$

where: $\bar{u} = \bar{i}\,u + \bar{j}\,v + \bar{k}\,w$

$$u = \frac{dx}{dt} \qquad v = \frac{dy}{dt} \qquad w = \frac{dz}{dt}$$

Euler method is based on selection of an immovable control volume *V* surrounded by a control surface *S*. Different fluid elements pass through this control volume. These elements are described by the different values of velocity, pressure, density etc. The values of these quantities in the selected points of the control volume are the subject of Euler's description.



$$\bar{u} = \bar{u}(x, y, z, t)$$

$$p = p(x, y, z, t)$$

$$\rho = \rho(x, y, z, t)$$

Leonhard Euler
1707 - 1783

where:

$$\bar{u} = \bar{i}\, u_x(x, y, z, t) + \bar{j}\, u_y(x, y, z, t) + \bar{k}\, u_z(x, y, z, t)$$

The material derivative

The material derivative is a particular interpretation of the complete derivative of a function of several variables, related to the Eulerian description of the fluid motion. It shows how an arbitrary flow parameter describing the fluid element changes with time when the element is moving in the field of this parameter. It is explained below using the example of an arbitrary scalar parameter H, which is a direct and involved function of time. If H is a function of Euler variables, then there is:

$$H = H\big(t, x(t), y(t), z(t)\big)$$

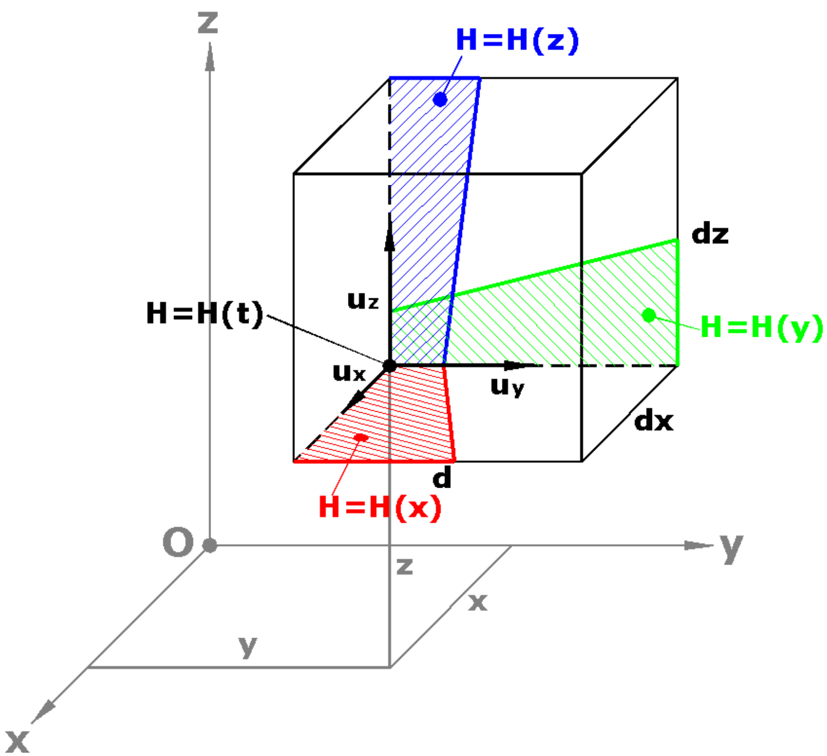
Following the definition of the complete differential there is:

$$\frac{DH}{Dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial y}\frac{dy}{dt} + \frac{\partial H}{\partial z}\frac{dz}{dt}$$

But: $\frac{dx}{dt} = u_x \quad \frac{dy}{dt} = u_y \quad \frac{dz}{dt} = u_z$ what leads to:

$$\frac{DH}{Dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x} u_x + \frac{\partial H}{\partial y} u_y + \frac{\partial H}{\partial z} u_z = \frac{\partial H}{\partial t} + \bar{u} \cdot \nabla H = \frac{\partial H}{\partial t} + \bar{u} \cdot gradH$$

Material derivative=local derivative+convective derivative



The local derivative shows the change of the parameter *H* with time at the point *(x, y, z)*, resulting from the unsteadiness of the field *H*.

The convective derivative shows the change of the parameter *H* with time, resulting from the motion of the fluid element with velocity $\bar{u}$ from the point with one value of *H* to the point with another value of *H*.

Application of the material derivative operator to the components of the velocity field enables calculation of the material acceleration, i.e. the acceleration of the fluid element moving in the unsteady and non-uniform field of flow:

$$\frac{Du_x}{Dt} = \frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} = a_x$$

$$\frac{Du_y}{Dt} = \frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} + u_z \frac{\partial u_y}{\partial z} = a_y$$

$$\frac{Du_z}{Dt} = \frac{\partial u_z}{\partial t} + u_x \frac{\partial u_z}{\partial x} + u_y \frac{\partial u_z}{\partial y} + u_z \frac{\partial u_z}{\partial z} = a_z$$

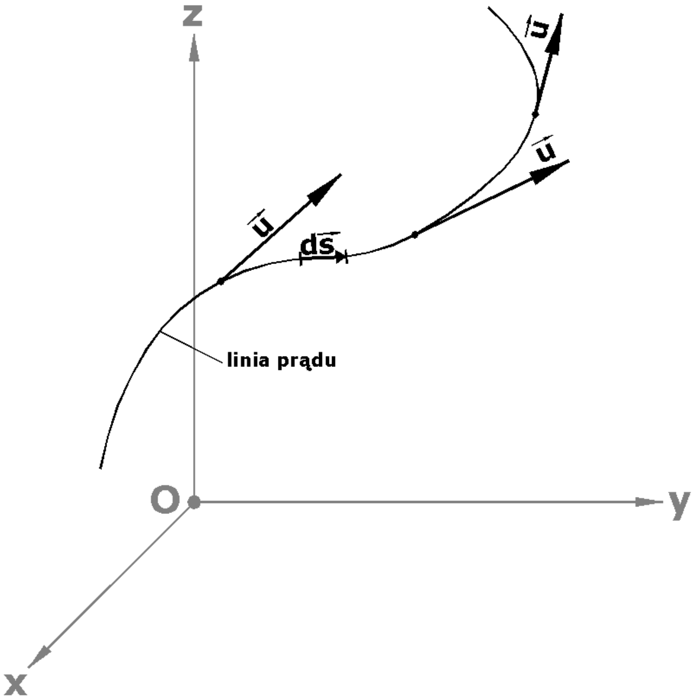
or in vector notation:

$$\frac{D\bar{u}}{Dt} = \frac{\partial \bar{u}}{\partial t} + \bar{u} \cdot grad\, \{ \bar{u} = \frac{\partial \bar{u}}{\partial t} + \left(\bar{u} \nabla \right) \bar{u} \text{¿}$$

Stream line is the line of the vector field of velocity, i.e. the line tangent to the velocity vector in every point of the velocity field in the given instant of time. If *ds* is the element of the stream line and *u* – the velocity vector, then there is:

$$d\bar{s} \times \bar{u} = 0$$

tangentiality condition

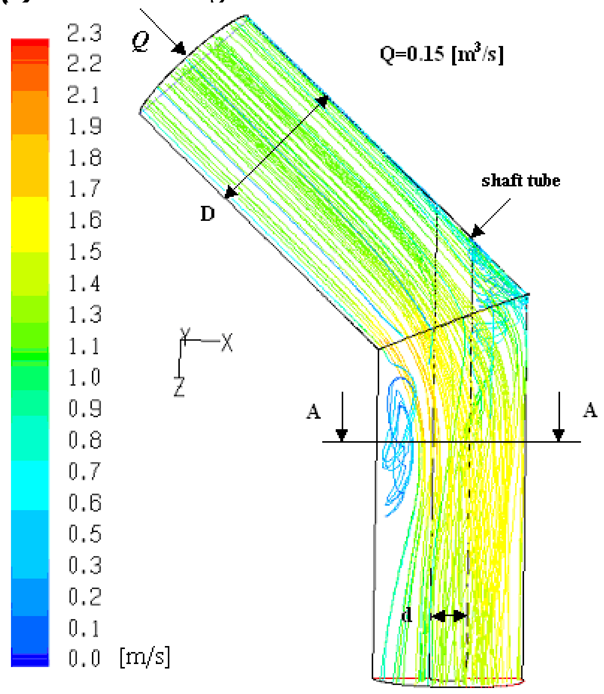


or:

$$u_z dy - u_y dz = 0$$

$$u_x dz - u_z dx = 0$$

$$u_y dx - u_x dy = 0$$



what leads to the stream line equation:

$$\frac{dx}{u_x} = \frac{dy}{u_y} = \frac{dz}{u_z}$$

In general only one, univocally determined, stream line passess through any point of the velocity field. If more stream lines converge in one point of the field then this is a **singular point**. If we draw stream lines through a line not being a stream line, we obtain a **stream surface**. If this line is a closed curve, we obtain a **stream tube**. If this tube has an infinitesimal cross-section, we obtain a **stream filament**. Stream tube is a good model of a real pipeline, for which we may determine:

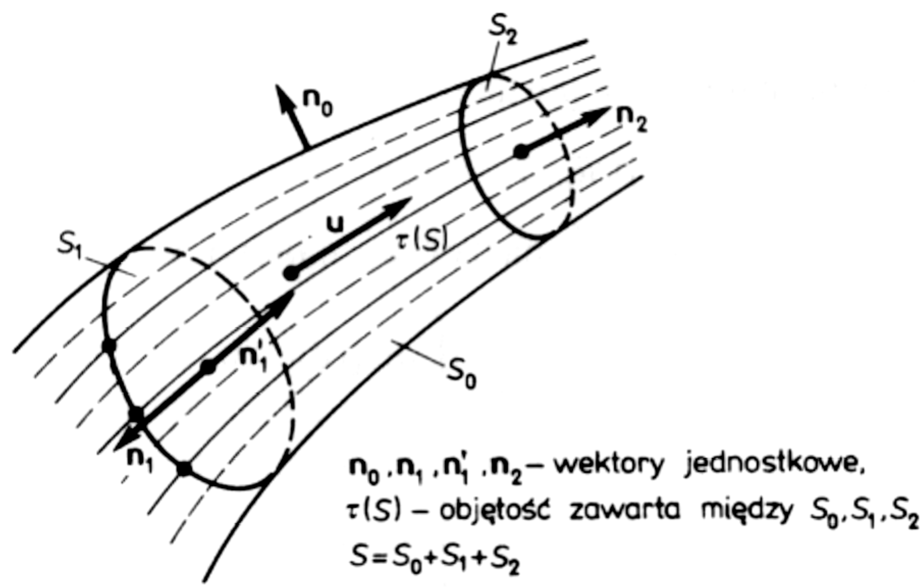
volumetric intensity of flow: $$Q = \int_S u_n \, dS$$

volumetric mean velocity: $$\tilde{u} = \frac{1}{S} \int_S u_n \, dS$$

mass intensity of flow: $$M = \int_S \rho u_n \, dS$$

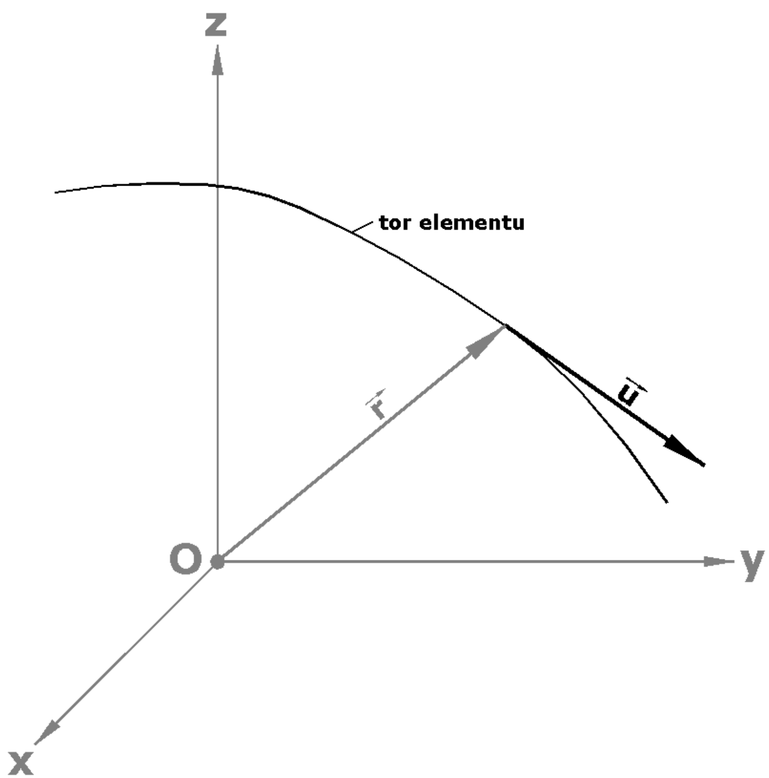
mass mean velocity: $$\tilde{u} = \frac{\int_S \rho u_n \, dS}{\int_S \rho \, dS}$$

a stream tube



where: u_n is the velocity component normal to the cross-section S

Path of the fluid element or **trajectory** is the geometrical location of the points in the field flow, through which the element passes in the consecutive instants of time.



Vector equation of the path:

$$\frac{d\bar{r}}{dt} = \bar{u}(\bar{r}, t)$$

In the scalar form:

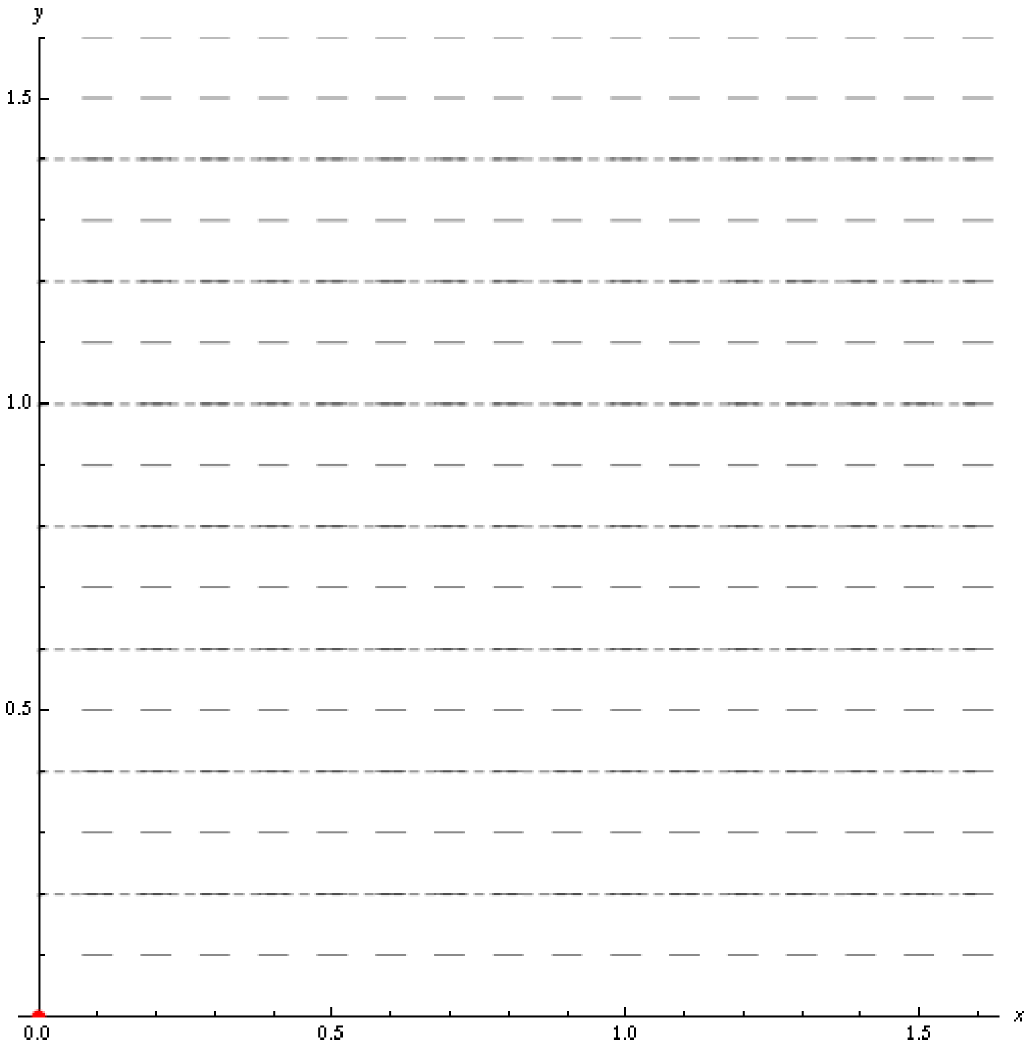
$$\frac{dx}{dt} = u_x(x, y, z, t) \qquad \frac{dy}{dt} = u_y(x, y, z, t)$$

$$\frac{dz}{dt} = u_z(x, y, z, t)$$

Solution requires taking into account the initial conditions for $t = t_0$

$$x(t) = x_0 \qquad y(t) = y_0 \qquad z(t) = z_0$$

In a general unsteady flow the stream lines, paths of the fluid elements and streak lines **do not coincide.**



Stream lines – grey colour

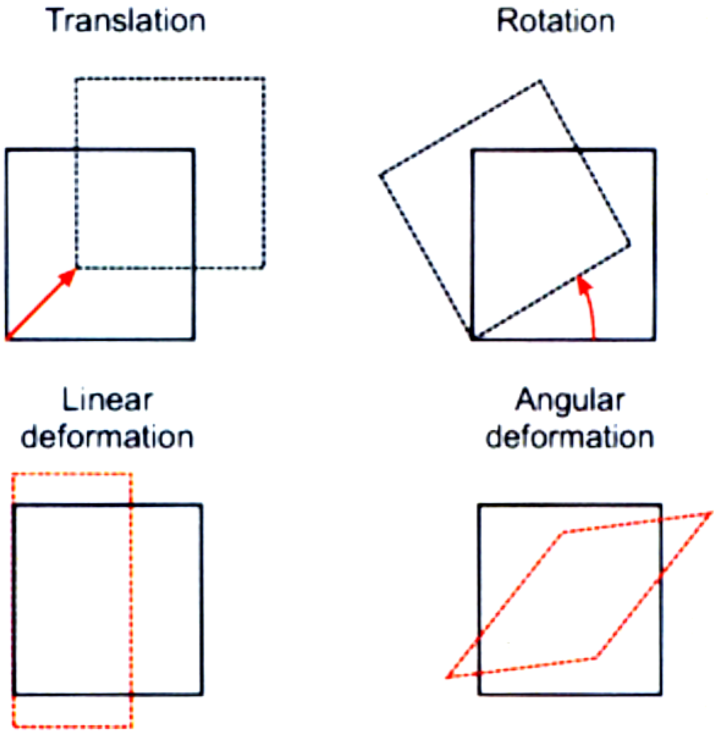
Paths of the elements – red colour

Streak lines – blue colour

Streak line is the trace of the fluid element drifting in the unsteady velocity field of the moving fluid.

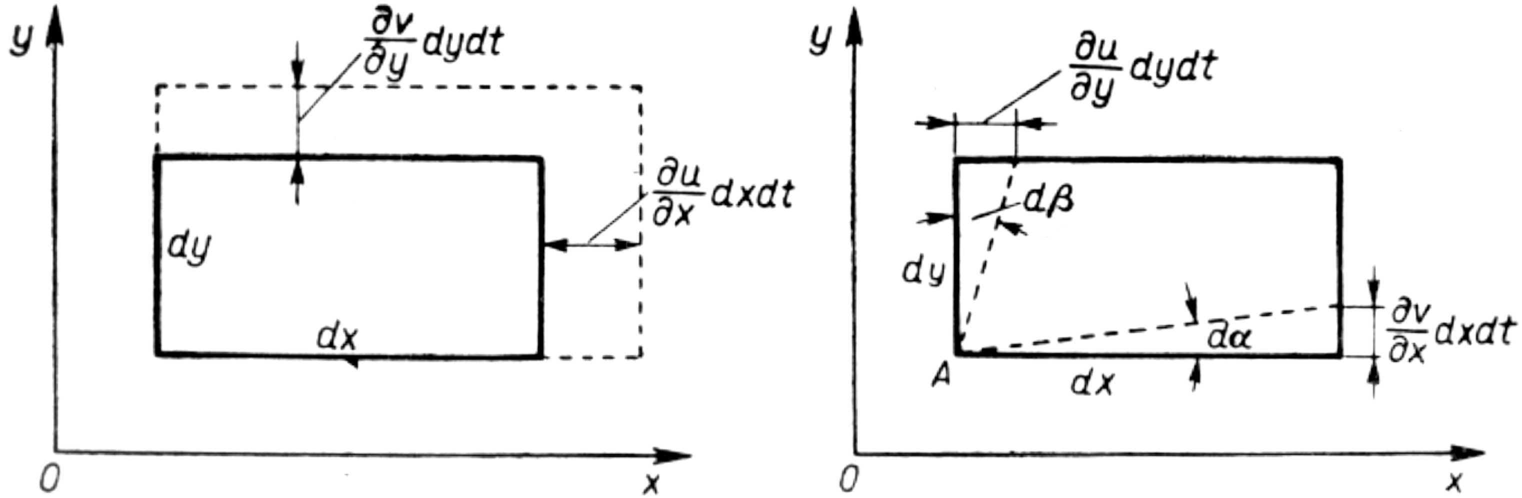
A general motion of the fluid element

A general motion of the rigid body may be considered as the sum of linear translation and rotation. As the fluids are not rigid, in their motion the deformation of the fluid elements must be additionally considered.



Thus the general motion of the fluid element may be treated as the superposition of the linear translation, rotation around the temporary centre and deformation. Deformation may be divided into linear deformation and angular (shearing) deformation.

Deformations in a two-dimensional case



Velocity of the fluid motion is: $\bar{u} = \bar{i}\,u + \bar{j}\,v$

The linear deformation of the fluid element takes place when the velocity component *u* varies in direction *x* and/or the velocity component *v* varies in direction *y* (left side of the picture). This leads to the increase in the element volume in time *dt* by:

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dxdydt$$

where the quantities in parantheses are the linear deformation velocities:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y}$$

The angular (shearing) deformation of the fluid element takes place when the velocity component *u* varies in the direction *y* and/or the velocity component *v* varies in the direction *x* (right side of the picture). This leads to rotation of the element walls by the angles:

$$d\alpha = \frac{\partial v}{\partial x} dt \qquad d\beta = \frac{\partial u}{\partial y} dt$$

The measure of the combined angular deformation is the expression:

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

The rigid rotation of the fluid element may be regarded as the sum of two deformations selected in such a way that the angles between the walls remain unchanged. The angular velocity of such a rotation may be written as:

$$\Omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

The symmetric tensor describing the deformation of the fluid element in three dimensions is named the **rate of strain tensor**:

$$[D] = \begin{vmatrix} \varepsilon_{xx}, \varepsilon_{xy}, \varepsilon_{xz} \\ \varepsilon_{yx}, \varepsilon_{yy}, \varepsilon_{yz} \\ \varepsilon_{zx}, \varepsilon_{zy}, \varepsilon_{zz} \end{vmatrix}$$

where the respective terms are described by the following relations:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \varepsilon_{yz} = \varepsilon_{zy} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} \qquad \varepsilon_{xz} = \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

Finally, a general motion of the fluid element may be described by the following relation:

$$\bar{u}_A = \bar{u}_0 + \bar{\omega}_0 \times \partial \bar{r} + \left[D \right]_0 \cdot \partial \bar{r}$$

The first Helmholtz theorem

The velocity of an arbitrary point of the fluid element may be composed of:

-translational velocity of the point selected as centre,

-rotational velocity around the axis passing through this centre (vector of this velocity defines the axis of rotation),

Hermann von Helmholtz
1821 - 1894

-deformation velocity of the fluid element.

In comparison with the analogical motion of a rigid body the following differences may be notoced:

- the formula for fluid is valid only close to the rotation centre,

-additional velocity of deformation is present in the fluid.

The closed system of equations of fluid mechanics

The equations, discussed in detail below, form the closed system of the fluid mechanics equations, which may be employed for description of realistic flows and for obtaining (through solution of these equations) information about the values of parameters describing these flows. The actual format of the system of equations depends on the adopted fluid and flow models.

Case No. 1: Incompressible fluid of constant viscosity

The closed system of equations is formed of:

- mass conservation equation $div\{\bar{u}=0$¿
- momentum conservation equation $$\rho \frac{D\bar{u}}{Dt} = \rho \bar{f} - grad p + \mu \Delta \bar{u}$$

These are equivalent to four scalar equations with four unknowns:

- pressure *p*
- velocity components u_x, u_y, u_z

In this case the temperature field does not influence the flow, but it depends itself on the velocity field through the entropy balance equation in the form:

$$\rho c\left(\frac{\partial T}{\partial t}+u_x\frac{\partial T}{\partial x}+u_y\frac{\partial T}{\partial y}+u_z\frac{\partial T}{\partial z}\right)=T\dot{s}_M+\lambda\Delta T$$

This form of entropy balance may be obtained from the original formula by substituting the relation for the fluid internal energy:

$$e=cT+e_0$$

In the case when the fluid viscosity depends on temperature, the balance of entropy equation is connected with the mass and momentum conservation equations through the relation: $\mu=\mu(T)$

Then we have the system of six equations with six unknowns:

- pressure p - velocity components u_x, u_y, u_z

- temperature T - viscosity coefficient μ

Case No. 2: Compressible fluid

In this case the closed system of equations is formed of:

- mass conservation equation $$\frac{\partial \rho}{\partial t}+div\left(\rho \bar{u}\right)=0$$

- momentum conservation equation $$\rho\frac{D\bar{u}}{Dt}=\rho\bar{f}-gradp-grad\left(\frac{2}{3}\mu div\{\bar{u}¿\right)+div\left(2\mu\left[D\right]\right)$$

- entropy balance equation $$\rho\frac{De}{Dt}=T\dot{s}_M+\frac{p}{\rho}\frac{Dp}{Dt}+\lambda\Delta T$$

- internal energy equation $$e=\int_{T_0}^{T}c_v\left(T\right)dT$$

- equation of state $$\frac{p}{\rho}=Z\left(p,T\right)RT$$

Z – compressibility function

R – gas constant

- additional relations $$\mu=\mu\left(T\right) \qquad c_V=c_V\left(T\right)$$

In this case we have the system of nine equations with nine unknowns:

- pressure p
- density ρ
- internal energy e
- temperature T
- viscosity coefficient μ
- velocity components u_x, u_y, u_z
- specific heat c_V

It is assumed that the thermal conductivity coefficient λ is constant and given.

Boundary and initial conditions

In order to enable solution of the above sytems of equations it is necessary to determine the appropriate boundary and (for unsteady flows) initial conditions. These conditions must be sufficient to enable determination of the arbitrary constants and arbitrary functions resulting from the integration of the above equations.

Mass conservation equation

Principle of mass conservation: in a closed physical system mass cannot be generated or annihilated.

Assumptions:

-we consider an unsteady three-dimensional flow of a compressible fluid,

-the fluid fills the space in a continuous way (no bubbles etc.),

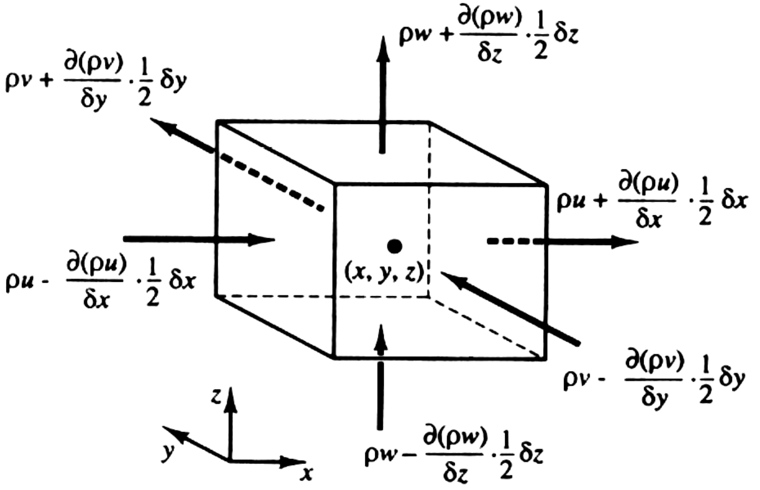
-we apply the Eulerian approach – a stationary control volume surrounded by a control surface.

With these assumptions the mass conservation principle reads:

the change of mass in the control volume = the flow of mass through the control surface

The change of mass in the control volume is equal to:

$$\frac{\partial}{\partial t}\left(\rho\delta x\delta y\delta z\right)=\frac{\partial \rho}{\partial t}\delta x\delta y\delta z$$



In turn the flow through the control surface is:

$$\left(\rho u-\frac{\partial\left(\rho u\right)}{\partial x}\frac{1}{2}\delta x\right)\delta y\delta z-\left(\rho u+\frac{\partial\left(\rho u\right)}{\partial x}\frac{1}{2}\delta x\right)\delta y\delta z+$$

$$+\left(\rho v-\frac{\partial\left(\rho v\right)}{\partial y}\frac{1}{2}\delta y\right)\delta x\delta z-\left(\rho v+\frac{\partial\left(\rho v\right)}{\partial y}\frac{1}{2}\delta y\right)\delta x\delta z+$$

$$+\left(\rho w-\frac{\partial\left(\rho w\right)}{\partial z}\frac{1}{2}\delta z\right)\delta x\delta y-\left(\rho w+\frac{\partial\left(\rho w\right)}{\partial z}\frac{1}{2}\delta z\right)\delta x\delta y$$

Equating both above expressions leads to (after dividing both sides by the control volume):

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=\frac{\partial \rho}{\partial t}+div(\rho \bar{u})=0$$

In the case of steady flow of a compressible fluid the mass conservation equation takes the form

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=div(\rho \bar{u})=0$$

In the case of steady flow of an incompressible fluid the mass conservation equation takes the form:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=div\{\bar{u}=0¿$$

In the case of a moving fluid element (Lagrange's description) the mass conservation equation takes the form:

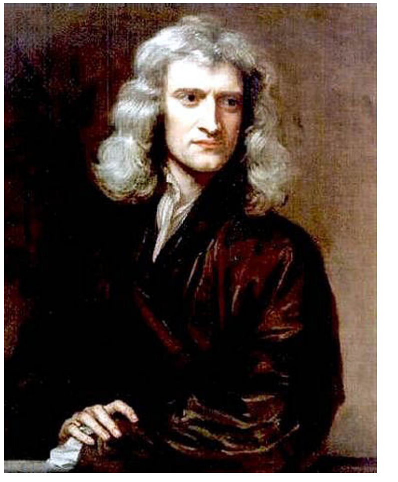
$$\frac{\partial \rho}{\partial t}+div(\rho \bar{u})=\frac{\partial \rho}{\partial t}+\bar{u}\cdot grad\,\rho+\rho\, div\{\bar{u}=\frac{D\rho}{Dt}+\rho\, div\{\bar{u}¿=0¿$$

Momentum conservation equation

The second law of Newton: the rate of change of momentum of a fluid element is equal to the sum of external forces acting on this element:

$$\frac{D\left(m\bar{u}\right)}{Dt}=\sum \bar{F}$$

Isaac Newton
1643 - 1727



The rate of change of momentum of the fluid element is defined by the material derivative of its velocity:

$$\rho\frac{Du}{Dt}=\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\frac{\partial\left(\rho u\right)}{\partial t}+div\left(\rho u\,\bar{u}\right)$$

$$\rho\frac{Dv}{Dt}=\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\frac{\partial\left(\rho v\right)}{\partial t}+div\left(\rho v\,\bar{u}\right)$$

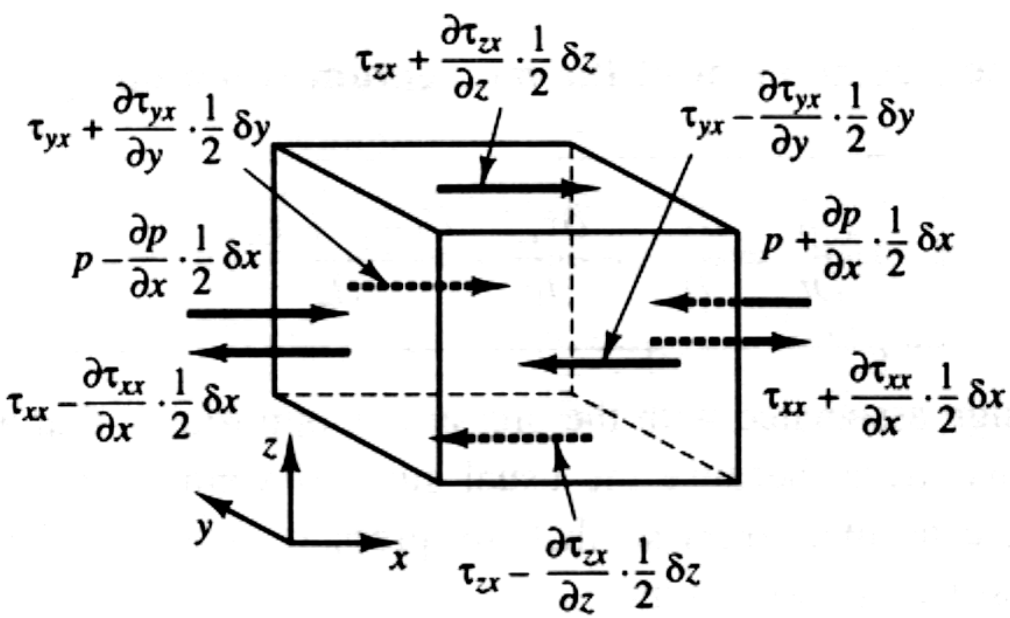
$$\rho\frac{Dw}{Dt}=\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\frac{\partial\left(\rho w\right)}{\partial t}+div\left(\rho w\,\bar{u}\right)$$

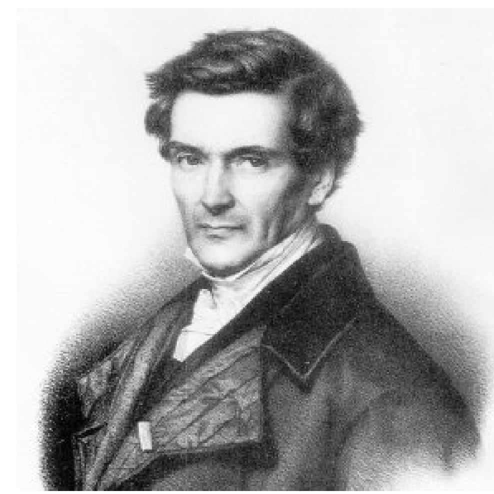
The right hand side is composed of the two categories of forces:

-surface forces (pressure forces and viscosity forces),

-mass forces (gravity forces, Coriolis forces, electromagnetic forces)

For example we will formulate the complete equation for the *x* direction, using the system of surface forces as in the picture:





Gaspard Coriolis 1792 - 1843

Forces acting on the element walls perpendicular to x direction

$$\left[\left(p-\frac{\partial p}{\partial x}\frac{1}{2}\delta x\right)-\left(\tau_{xx}-\frac{\partial \tau_{xx}}{\partial x}\frac{1}{2}\delta x\right)\right]\delta y\delta z+\left[-\left(p+\frac{\partial p}{\partial x}\frac{1}{2}\delta x\right)+\left(\tau_{xx}+\frac{\partial \tau_{xx}}{\partial x}\frac{1}{2}\delta x\right)\right]\delta y\delta z=$$

$$=\left(-\frac{\partial p}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}\right)\delta x\delta y\,\partial z$$

Forces acting on the element walls perpendicular to y direction

$$-\left(\tau_{yx}-\frac{\partial \tau_{yx}}{\partial y}\frac{1}{2}\delta y\right)\delta x\delta z+\left(\tau_{yx}+\frac{\partial \tau_{yx}}{\partial y}\frac{1}{2}\delta y\right)\delta x\delta z=\frac{\partial \tau_{yx}}{\partial y}\delta x\delta y\delta z$$

Forces acting on the element walls perpendicular to z direction

$$-\left(\tau_{zx}-\frac{\partial \tau_{zx}}{\partial z}\frac{1}{2}\delta z\right)\delta x\delta y+\left(\tau_{zx}+\frac{\partial \tau_{zx}}{\partial z}\frac{1}{2}\delta z\right)\delta x\delta y=\frac{\partial \tau_{zx}}{\partial z}\delta x\delta y\delta z$$

After adding the above expressions together and dividing by the element volume we obtain the surface forces acting in direction *x*

$$\frac{\partial\left(-p+\tau_{xx}\right)}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}$$

After supplementing the expression with the unit mass force f and substituting it to the initial formula we obtain:

$$\rho\frac{Du}{Dt}=\rho f_x+\frac{\partial\left(-p+\tau_{xx}\right)}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}$$

and analogically for the remaining two directions:

$$\rho\frac{Dv}{Dt}=\rho f_y+\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial\left(-p+\tau_{yy}\right)}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}$$

$$\rho\frac{Dw}{Dt}=\rho f_z+\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial\left(-p+\tau_{zz}\right)}{\partial z}$$

Stress tensor in fluid

$$[P] = \begin{bmatrix} -p+\tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & -p+\tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & -p+\tau_{zz} \end{bmatrix}$$

State of stress in the fluid

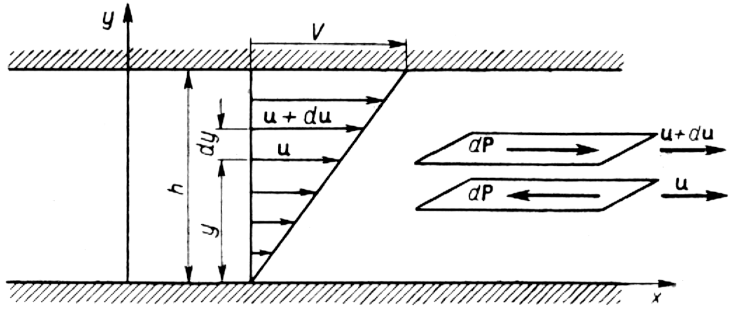
It may be proved that <u>the tensor of stress in the fluid is symmetrical</u> i.e.: $\tau_{xy} = \tau_{yx}$ etc.

This reduces the number of unknown viscous stresses to 6, which must be determined on the basis of the selected model of fluid. In most cases the <u>Newtonian model</u> of fluid is employed.

The Newtonian model of fluid is based on the following assumptions:

-the fluid is isotropic, i.e. it has the same properties in all directions,

-the stresses in the fluid are linear functions of the rate of strain.



$$\tau_{yx} = \mu \frac{\partial u}{\partial y}$$

where:

μ - the dynamic viscosity coefficient

In the three-dimensional flow of a compressible fluid the Newtonian fluid model is described by the following relations:

$$\tau_{xx}=2\mu\frac{\partial u}{\partial x}+\lambda\, div\{\bar{u}\text{¿}$$

$$\tau_{yy}=2\mu\frac{\partial v}{\partial y}+\lambda\, div\{\bar{u}\text{¿}$$

$$\tau_{zz}=2\mu\frac{\partial w}{\partial z}+\lambda\, div\{\bar{u}\text{¿}$$

$$\tau_{xy}=\tau_{yx}=\mu\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)$$

$$\tau_{yz}=\tau_{zy}=\mu\left(\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\right)$$

$$\tau_{xz}=\tau_{zx}=\mu\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)$$

where:

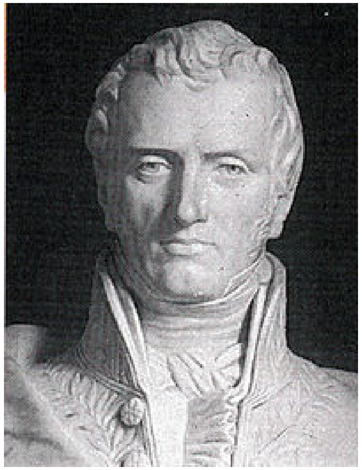
$$div\{\bar{u}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\text{¿}$$

λ – volumetric viscosity coefficient

According to Stokes hypothesis:

$$\lambda=-\frac{2}{3}\mu$$

In an incompressible fluid $div\{\bar{u}=0\text{¿}$ hence the second terms of the normal stresses are reduced to zero.



Claude Navier
1785 - 1836

Navier-Stokes equation

Substitution of the relations resulting from the Newtonian fluid model into the equations of conservation of the fluid momentum leads to the **Navier-Stokes equation.**

George Stokes
1819 - 1903

This equation may be written in the form of three scalar equations:

$$\rho\frac{Du}{Dt}=\rho f_x-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x}+\lambda\, div\{\bar{u}\text{¿}\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]$$

$$\rho\frac{Dv}{Dt}=\rho f_y-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[2\mu\frac{\partial v}{\partial y}+\lambda\, div\{\bar{u}\text{¿}\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]$$

$$\rho\frac{Dw}{Dt}=\rho f_z-\frac{\partial p}{\partial z}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]+\frac{\partial}{\partial z}\left[2\mu\frac{\partial w}{\partial z}+\lambda\, div\{\bar{u}\text{¿}\right]$$

In the vector form the Navier-Stokes equation reads:

$$\rho \frac{D\bar{u}}{Dt} = \rho \bar{f} - grad p + grad\left(\lambda div \{ \bar{u} ¿ \right) + div\left(2\mu [D]\right)$$

$$A = B + C + D + E$$

A – rate of change of momentum of the fluid element

B- mass force

C- surface pressure force

D – surface force connected with fluid viscosity and resulting from the change of volume of the compressible fluid element (compression or expansion)

E- surface force connected with fluid viscosity and resulting from the linear and shearing deformation of the fluid element

In an incompressible fluid the Navier-Stokes equation simplifies to the form:

$$\rho \frac{D\bar{u}}{Dt} = \rho \bar{f} - gradp + div\left(2\mu\left[D\right]\right)$$

If additionally a constant fluid viscosity is assumed, we obtain:

$$\rho \frac{D\bar{u}}{Dt} = \rho \bar{f} - gradp + \mu \Delta \bar{u}$$

Further possible simplification is the assumption of zero viscosity of the fluid, which leads to the **Euler equation**, describing the motion of an incompressible and inviscid fluid:

$$\rho \frac{D\bar{u}}{Dt} = \rho \bar{f} - gradp$$

The Navier-Stokes equation may be solved analytically only for a few simplified cases. However, it forms the basis of contemporary Computational Fluid Dynamics.

Energy conservation equation

Kinetic energy of the fluid may be treated as the sum of the macroscopic motion energy and the molecular motion (or internal) energy:

$$\int_V \left(\frac{u^2}{2} + e \right) dV \qquad \bar{u}\left(u_x, u_y, u_z\right) \qquad u = \left|\bar{u}\right|$$

The rate of change (i.e. material derivative) of the total kinetic energy of the fluid volume *V* surrounded by the surface *S* is equal to the sum of the power of mass forces, the power of surface forces and the stream of energy (heat) supplied to the fluid volume.

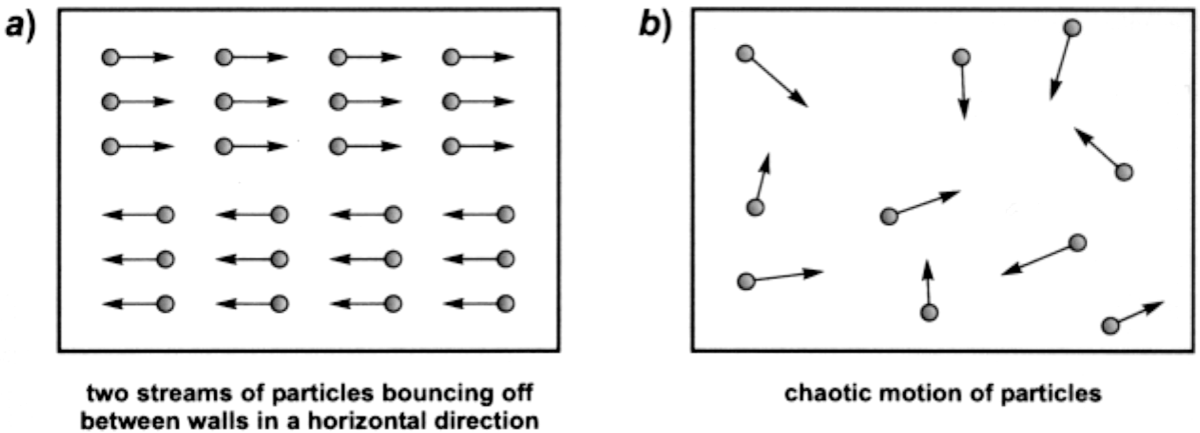
$$\frac{D}{Dt} \int_V \rho \left(\frac{u^2}{2} + e \right) dV = \int_V \rho \bar{f} \cdot \bar{u} \, dV + \int_{S(V)} \bar{\tau} \cdot \bar{u} \, dS - \int_{S(V)} \bar{j} \cdot \bar{n} \, dS$$

where:

- $\bar{f}$ unit mass force $\bar{f}\left(f_x, f_y, f_z\right)$
- $\bar{\tau}$ unit surface force
- $\bar{j}$ stream of supplied energy (heat) $\bar{j}\left(j_x, j_y, j_z\right)$
- $\bar{n}$ external unit length normal vector

Balance of entropy equation

Entropy S is a function of state parameters (such as temperature, pressure etc.) of the fluid and it is the measure of chaos in molecular motion and the measure of „useless” energy of a given system.



a – system with low entropy

b – system with high entropy

Unit of entropy S - $\left[\frac{J}{K}\right]$

Unit of specific entropy s - $\left[\frac{J}{kg \cdot K}\right]$

Ice melting in a glass is an example of increasing entropy

Entropy s is transported with heat according to the Clausius formula:

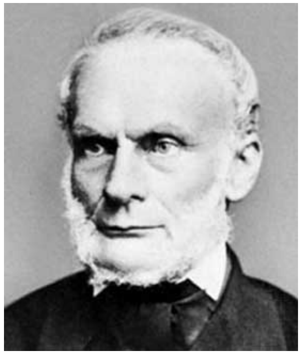
$$j_s = \frac{1}{T} j$$

where: j_s stream of entropy

j stream of heat

T temperature at which transport takes place

Rudolf Clausius
1822 - 1888



Entropy changes with the fluid state parameters (Gibbs formula):

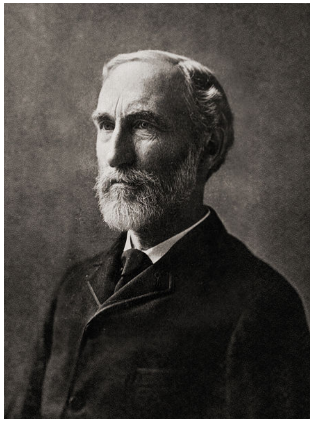
$$T \frac{Ds}{Dt} = \frac{De}{Dt} + p \frac{D}{Dt}\left(\frac{1}{\rho}\right)$$

where: p - pressure

e – fluid internal energy

ρ - fluid density

Josiah Gibbs
1839 - 1903



The second law of thermodynamics: in any real process the sum of changes of entropy of all bodies taking part in the process is always positive.

The rate of change (i.e. the material derivative) of entropy in the fluid volume *V(S)* is equal to the production of entropy inside this volume and the stream of entropy through the fluid surface *S*.

$$\frac{D}{Dt}\int_V \rho s dV = \int_V \dot{s}\, dV - \int_{S(V)} \bar{j}_s \cdot \bar{n}\, dS$$

where: $\dot{s}$ volumetric intensity of the entropy sources

The above equation may be converted into the form of a single volumetric integral:

$$\int_V \left(\rho \frac{Ds}{Dt} - \dot{s} + div \frac{\bar{j}}{T} \right) dV = 0$$

As the fluid volume *V* was arbitrarily selected, the function under the integral must also be zero, leading to the balance of entropy equation in the differential form (i.e. for a fluid element):

$$\rho \frac{Ds}{Dt} = \dot{s} - div \frac{\bar{j}}{T}$$

By using the relation of Gibbs we may obtain:

$$\dot{s} = \frac{\rho}{T}\frac{De}{Dt} - \frac{p}{\rho T}\frac{D\rho}{Dt} + div\,\frac{\bar{j}}{T} \qquad \text{or:} \qquad \rho\frac{De}{Dt} = T\,\dot{s}_M + \frac{p}{\rho}\frac{D\rho}{Dt} + \lambda \Delta T$$

The above equation may be re-formulated in the following way, using the conservation equations of mass, momentum and energy, together with the thermal conductivity law of Fourier:

$$\dot{s} = \dot{s}_M + \dot{s}_T = \frac{\mu}{T}\Big[\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)^2 + \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)^2 + \left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right)^2 + \frac{2}{3}\left(\frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y}\right)^2 + \frac{2}{3}\left(\frac{\partial u_x}{\partial x} - \frac{\partial u_z}{\partial z}\right)^2 + \frac{2}{3}\left(\frac{\partial u_y}{\partial y} - \frac{\partial u_z}{\partial z}\right)^2\Big] + \frac{\lambda}{T^2}\left(gradT\right)^2$$

The balance of entropy equation in the above form describes the process of continouous dissipation of mechanical energy of the flowing fluid and conversion of this energy into heat.

Law of Fourier -> $\bar{j} = -\lambda\, gradT$

Joseph Fourier 1768 - 1830

Bernoulli equation

Daniel Bernoulli
1700 - 1782

Bernoulli equation expresses, under certain assumptions, the principles of momentum conservation and energy conservation of the fluid.

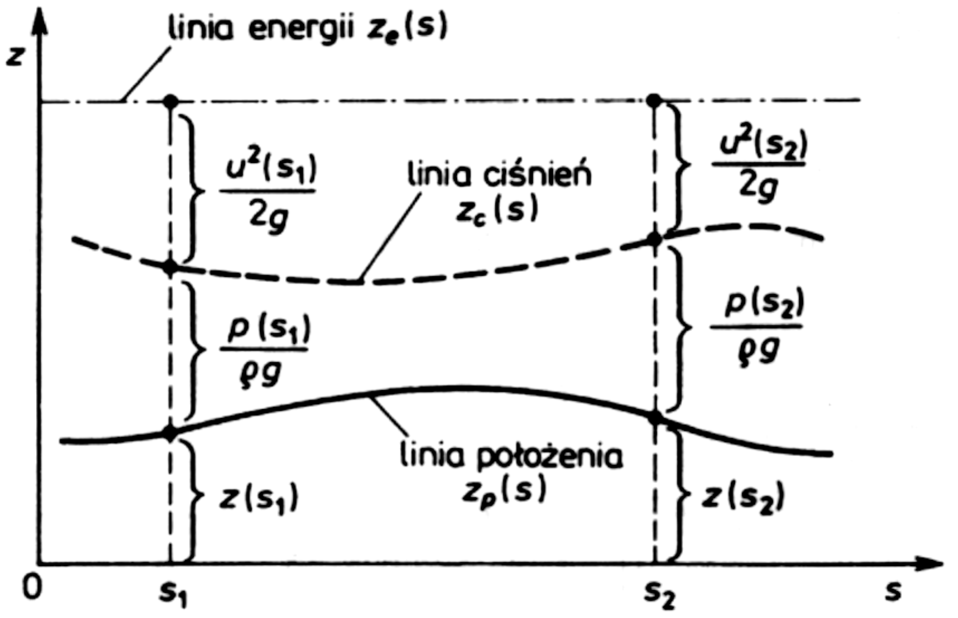
Assumptions:

- the flow is stationary $\frac{\partial}{\partial t}=0$
- the fluid is inviscid $\mu=0$
- the fluid is barotropic $\rho=\rho(p)$
- The mass forces form a potential field $\bar{f}=-grad\,\Pi$

Under such assumptions the Euler equation may be integrated:

$$\rho\frac{D\bar{u}}{Dt}=\rho\bar{f}-grad p$$

Bernoulli equation (1738)



$$gz + \frac{p}{\rho} + \frac{u^2}{2} = const$$

or

$$z + \frac{p}{\rho g} + \frac{u^2}{2g} = const$$

The sum of the potential energy of the mass forces, the pressure energy and the kinetic energy of the fluid is constant.

or:

The sum of the geometrical elevation *z*, the pressure head (i.e. the height to which the fluid is elevated under pressure *p*) and the velocity head (i.e. the height from which the falling fluid element achieves velocity *u*) is constant.

Other forms of the Bernoulli equation are possible if particular forms of the barotropic relation are adopted. For example, in the case of a gas undergoing an adiabatic process this relation reads:

$$\rho = \frac{\rho_0}{p_0^{1/\kappa}} p^{1/\kappa}$$

where κ is the Poisson adiabatic exponent $\kappa = \frac{c_p}{c_v}$

Then the Bernoulli equation takes the form:

$$\frac{u^2}{2} + \frac{\kappa}{\kappa - 1} \frac{p_0}{\rho_0} \left[\left(\frac{p}{p_0} \right)^{(\kappa-1)/\kappa} - 1 \right] + gz = const$$

Simeon Poisson
1781 - 1840

Comparison of the Bernoulli equation development with the energy conservation equation for a stream tube shows that, with disregarding the fluid internal energy *e* and the thermal conductivity of the fluid, the Bernoulli equation describes the energy conservation principle as well.