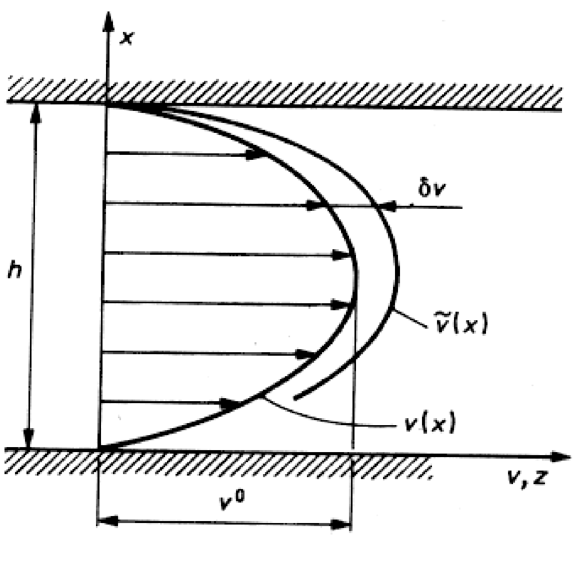
J. Szantyr – Lecture No. 11 – Computations of Viscous Flows – Finite Element Method

The analysed flow domain is divided into many small parts, so called finite elements. In the selected points of the elements (nodes) the values of the desired function, e.g. velocity, pressure etc. are to be determined. The solution is sought in the form of a **basis function** approximating the solution. Parameters of the approximating function are determined by means of the **Variational Calculus.**

Example of the one-dimensional solution – 2D Poiseuille' flow.



The equation describing the flow:

$$L = \frac{\Delta p}{\Delta z} + \mu \frac{d^2 \upsilon}{dx^2} = 0$$

If $\tilde{\upsilon}$ is the approximate solution, then in general: $L(\tilde{\upsilon}) \neq 0$ and the measure of the error is the weighed residual:

$$R = \int_0^h L(\tilde{\upsilon}) W dx = \int_0^h L(\tilde{\upsilon}) \tilde{\upsilon} dx$$

The condition for minimization of the functional R has the form:

$$\delta R = \int_0^h L(\tilde{\upsilon})\delta\tilde{\upsilon}dx = \int_0^h \left(\frac{\Delta p}{\Delta z}\delta\tilde{\upsilon} - \mu \frac{d\tilde{\upsilon}}{dx}\frac{d\delta\tilde{\upsilon}}{dx} \right)dx = 0$$

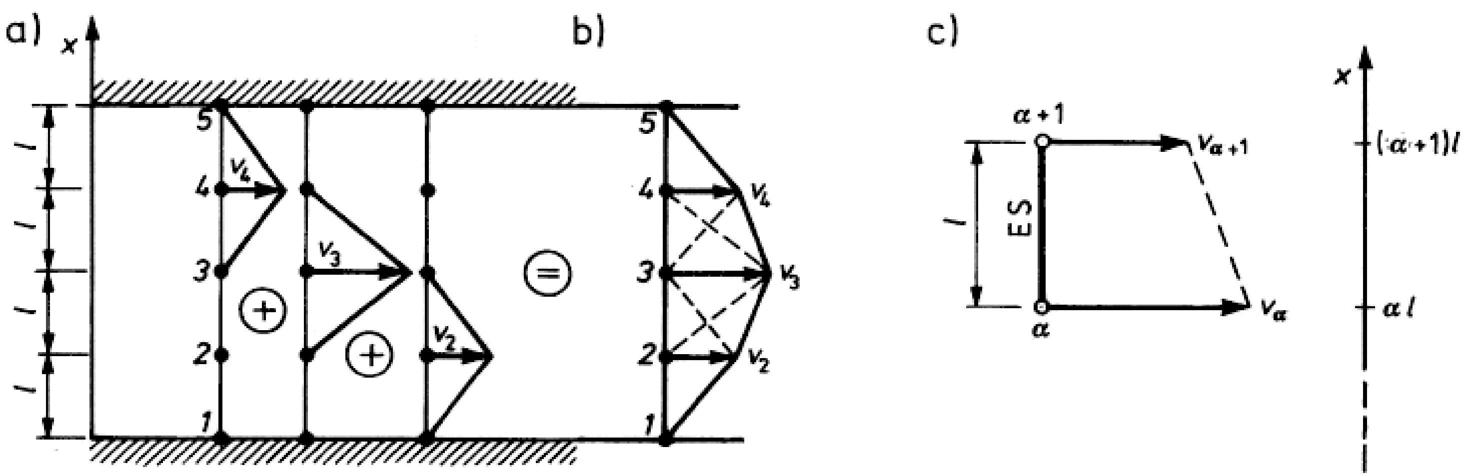
This can be written as: $$\mu \int_0^h \frac{d\tilde{\upsilon}}{dx}\frac{d\delta\tilde{\upsilon}}{dx}dx = \frac{\Delta p}{\Delta z}\int_0^h \delta\tilde{\upsilon}dx$$

The interval of integration [0,h] is divided into 4 equal sections (finite elements) having lengths of l=h/4. Their endpoints determine 5 nodes. In each of them there is the nodal value of velocity:

$$\upsilon_\alpha = \upsilon((\alpha - 1)/l) \rightarrow \alpha = 2,3,4$$

In the first and last node there are boundary values:

$$\upsilon_1 = \upsilon_5 = 0$$



The sought solution is approximated by the local linear basis functions, defined within each finite element:

$$\tilde{\upsilon}_\alpha = \upsilon_{\alpha-1}(1-x') + \upsilon_\alpha x' \qquad \text{where:} \quad x' = x/l \qquad x' \in [0,1]$$

After substituting the local basis functions the condition for minimalization of the functional R takes the form:

$$\frac{\mu}{l}\sum_1^4 \int_0^1 \frac{d\tilde{\upsilon}}{dx'}\frac{d\delta\tilde{\upsilon}}{dx'}dx' = l\frac{\Delta p}{\Delta z}\sum_1^4 \int_0^1 \delta\tilde{\upsilon}dx'$$

which may be developed into:

$$\frac{\mu}{l}\int_0^1 \left\langle \left[\frac{d}{dx'}(\upsilon_2 x')\frac{d}{dx'}(\delta\upsilon_2 x')\right] + \left[\frac{d}{dx}(\upsilon_3 x' + \upsilon_2(1-x'))\frac{d}{dx'}(\delta\upsilon_3 x' + \delta\upsilon_2(1-x'))\right]\right.$$

$$\left. + \left[\frac{d}{dx'}(\upsilon_4 x' + \upsilon_3(1-x'))\frac{d}{dx'}(\delta\upsilon_4 x' + \delta\upsilon_3(1-x'))\right] + \left[\frac{d}{dx'}(\upsilon_4(1-x'))\frac{d}{dx'}(\delta\upsilon_4(1-x'))\right] \right\rangle dx =$$

$$= l\frac{\Delta p}{\Delta z}\int_0^1 \{\delta(\upsilon_2 x') + \delta(\upsilon_3 x' + \upsilon_2(1-x')) + \delta(\upsilon_4 x' + \upsilon_3(1-x')) + \delta(\upsilon_4(1-x'))\}$$

After integration and re-formulation it leads to the system of linear equations for the unknown nodal values of velocity:

$$\frac{\mu}{cl^2}\left[(2v_2 - v_3)\delta v_2 + (-v_2 + 2v_3 - v_4)\delta v_3 + (-v_3 + 2v_4)\delta v_4\right] = \delta v_2 + \delta v_3 + \delta v_4$$

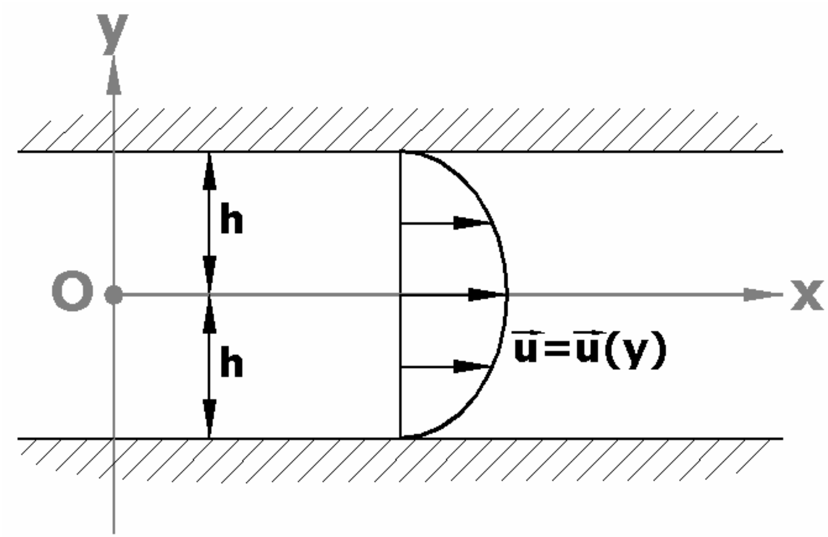
$$\frac{\mu}{cl^2}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

where: $$c = \frac{\Delta p}{\Delta z}$$

Solution of the system produces the following results:

$$\upsilon_2 = \upsilon_4 = \frac{3}{32}\frac{\Delta p}{\Delta z}\frac{h^2}{\mu} \qquad \upsilon_3 = \frac{4}{32}\frac{\Delta p}{\Delta z}\frac{h^2}{\mu}$$

The analytical solution: steady laminar flow between two parallel infinite flat plates (Poiseuille flow)



Given: $\frac{\Delta p}{\Delta x} = const$

Boundary conditions:

$u=0$ for $y=h$

$u=0$ for $y=-h$

The Navier-Stokes equation takes the form: $$\frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{\Delta p}{\Delta x}$$

After double integration we obtain:

$$u(y) = \frac{1}{\mu}\frac{\Delta p}{\Delta x}\frac{y^2}{2} + C_1 y + C_2$$

The integration constants are determined from the boundary condtions:

$$u(y) = -\frac{1}{2\mu}\frac{\Delta p}{\Delta x}\left(h^2 - y^2\right)$$

Transformation to the notation adopted in the Finite Element solution leads to:

$$y = x - \frac{H}{2} \qquad h = \frac{H}{2} \qquad \upsilon(x) = \frac{H^2}{2\mu}\frac{\Delta p}{\Delta z}\left(\frac{x}{H} - \frac{x^2}{H^2}\right)$$

$$\upsilon\left(\frac{H}{4}\right) = \frac{H^2}{2\mu}\frac{\Delta p}{\Delta z}\left(\frac{H}{4H} - \frac{H^2}{16H^2}\right) = \frac{3}{32}\frac{\Delta p}{\Delta z}\frac{H^2}{\mu}$$

$$\upsilon\left(\frac{3H}{4}\right) = \frac{H^2}{2\mu}\frac{\Delta p}{\Delta z}\left(\frac{3H}{4H} - \frac{9H^2}{16H^2}\right) = \frac{3}{32}\frac{\Delta p}{\Delta z}\frac{H^2}{\mu}$$

$$\upsilon\left(\frac{H}{2}\right) = \frac{H^2}{2\mu}\frac{\Delta p}{\Delta z}\left(\frac{H}{2H} - \frac{H^2}{4H^2}\right) = \frac{4}{32}\frac{\Delta p}{\Delta z}\frac{H^2}{\mu}$$

Hence the analytical solution is exactly equal to the numerical solution using FEM, but only in the nodes.