J. Szantyr – Lecture No. 13 – Theoretical Principles and Modelling of Turbulence

Mathematical description of the turbulent motion of fluids is done by means of **Reynolds equations**. Reynolds has assumed that in the turbulent flow all characteristic parameters, including velocity and pressure of the fluid, may be presented in the form of sums of their mean values (more precisely: slowly varying values) and turbulent fluctuations, i.e.:

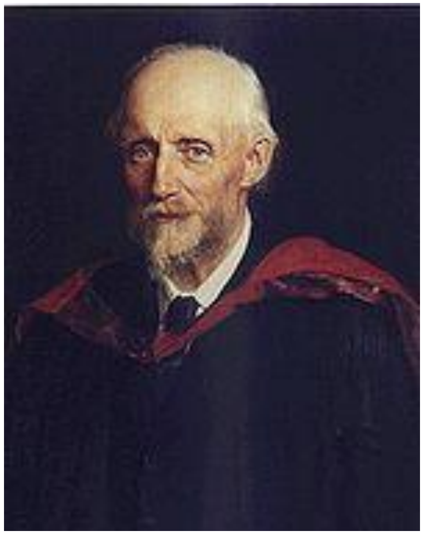
$$\overline{u} = \overline{U} + \overline{u}' \qquad p = P + p'$$

where U is the mean velocity of flow

$$\overline{U} = \overline{i}U + \overline{j}V + \overline{k}W$$

and u' is the turbulent fluctuation of velocity

$$\overline{u}' = \overline{i}u' + \overline{j}v' + \overline{k}w'$$



Osborne Reynolds
1842 - 1912

Substitution of so defined velocities and pressure into the Navier-Stokes equation leads to the explicit appearance of new surface forces, called the **turbulent stresses:**

$$\rho\frac{DU}{Dt}=\rho f_x-\frac{\partial P}{\partial x}+\mu divgradU+\rho\left[-\frac{\partial\widetilde{u'^2}}{\partial x}-\frac{\partial\widetilde{u'v'}}{\partial y}-\frac{\partial\widetilde{u'w'}}{\partial z}\right]$$

$$\rho\frac{DV}{Dt}=\rho f_y-\frac{\partial P}{\partial y}+\mu divgradV+\rho\left[-\frac{\partial\widetilde{u'v'}}{\partial x}-\frac{\partial\widetilde{v'^2}}{\partial y}-\frac{\partial\widetilde{v'w'}}{\partial z}\right]$$

$$\rho\frac{DW}{Dt}=\rho f_z-\frac{\partial P}{\partial z}+\mu divgradW+\rho\left[-\frac{\partial\widetilde{u'w'}}{\partial x}-\frac{\partial\widetilde{v'w'}}{\partial y}-\frac{\partial\widetilde{w'^2}}{\partial z}\right]$$

The above equations describe the flow of an <u>incompressible fluid</u>

Normal stresses:

$$\tau_{xx} = -\rho \widetilde{u'^2} \qquad \tau_{yy} = -\rho \widetilde{v'^2} \qquad \tau_{zz} = -\rho \widetilde{w'^2}$$

Tangential (shear) stresses:

$$\tau_{xy} = \tau_{yx} = -\rho \widetilde{u'} \widetilde{v'}$$

$$\tau_{xz} = \tau_{zx} = -\rho \widetilde{u'} \widetilde{w'} \qquad \tau_{yz} = \tau_{zy} = -\rho \widetilde{v'} \widetilde{w'}$$

<u>The turbulent stresses, also known as Reynolds stresses, depend on the values of turbulent fluctuations of the flow velocity, not on the fluid viscosity.</u> It may be shown that they form a symmetric stress tensor. They constitute additional 6 unknowns in the Reynolds equation describing the turbulent flow. In order to reduce the number of unknowns and close the system of equations the appropriate **turbulence models** must be introduced**.** <u>Reynolds equation forms the basis of the majority of commercial computer codes used in Computational Fluid Dynamics (CFD).</u>

The Boussinesq hypothesis (1877)

Boussinesq has assumed that the turbulent Reynolds stresses may be related to the tensor of mean rates of strain in the fluid in the similar way as the Newton fluid model relates the viscous stresses to this tensor. The difference is that in the case of turbulent stresses the proportionality coefficient is called the dynamic turbulent viscosity coefficient μ_t .This coefficient is not the physical characteristic of the fluid, but the characteristic of the flow depending on turbulence.

$$\tau_{ij} = -\rho u_i' u_j' = \mu_t \left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i} \right)$$

The Boussinesq hypothesis has a rational basis, because in the regions of high gradients of mean velocities the process of turbulence generation is the most intensive.

Joseph Boussinesq
1842 - 1929

The symmetric rate of strain tensor, describing the deformation of a fluid element, has the following form:

$$[D] = \begin{vmatrix} \varepsilon_{xx}, \varepsilon_{xy}, \varepsilon_{xz} \\ \varepsilon_{yx}, \varepsilon_{yy}, \varepsilon_{yz} \\ \varepsilon_{zx}, \varepsilon_{zy}, \varepsilon_{zz} \end{vmatrix}$$

Where the respective components are described by the relations:

$$\varepsilon_{xx} = \frac{\partial U}{\partial x} \qquad \varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2}\left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\right)$$

$$\varepsilon_{yy} = \frac{\partial V}{\partial y} \qquad \varepsilon_{yz} = \varepsilon_{zy} = \frac{1}{2}\left(\frac{\partial W}{\partial y} + \frac{\partial V}{\partial z}\right)$$

$$\varepsilon_{zz} = \frac{\partial W}{\partial z} \qquad \varepsilon_{xz} = \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial U}{\partial z} + \frac{\partial W}{\partial x}\right)$$

Boussinesq has assumed that the turbulent viscosity coefficient is a scalar quantity, while the turbulent state of stress in the fluid is strongly asymmetrical and in order to provide an exact description **the turbulent viscosity coefficient should be a tensor.**

It may be said that the Boussinesq hypotheis creates a new fluid model – **the turbulent fluid.**

A general principle of formation of the turbulence models

The majority of turbulence models makes use of the Boussinesq hypothesis. Then the objective of modelling is determination of the turbulent viscosity coefficient. The value of this coefficient may be determined in the form:

$$\nu_t = \frac{\mu_t}{\rho} = f\left(y_1, y_2, y_3, \ldots . y_n\right)$$

The objective of modelling is the form of the function f as well as the relations determining the values of arguments y in the required points of the flow domain. Depending on the number of these relations we may describe the corresponding turbulence models as zero-equation, one-equation, two-equation etc.

Zero-equation model – Prandtl's mixing length

Mixing length originally was regarded as the hypothetical distance, at which the exchange of momentum due to turbulent stresses between neighbouring fluid elements is completed. Prandtl regarded it as the analogue of the mean free path in gases. Now we interprete it as the mean characteristic of a turbulent mixing of fluid.

In the two-dimensional flow we have:

$$\tau_{xy} = \rho \cdot l_m^2 \cdot \left|\frac{\partial U}{\partial y}\right| \cdot \frac{\partial U}{\partial y}$$

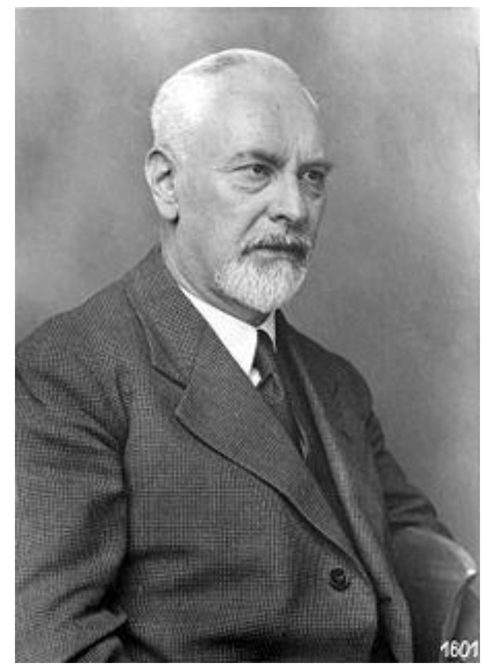
Shear stress

Then:

$$\nu_t = l_m^2 \cdot \left|\frac{\partial U}{\partial y}\right|$$

Kinematic coefficient of turbulent viscosity

Where: l_m - mixing length

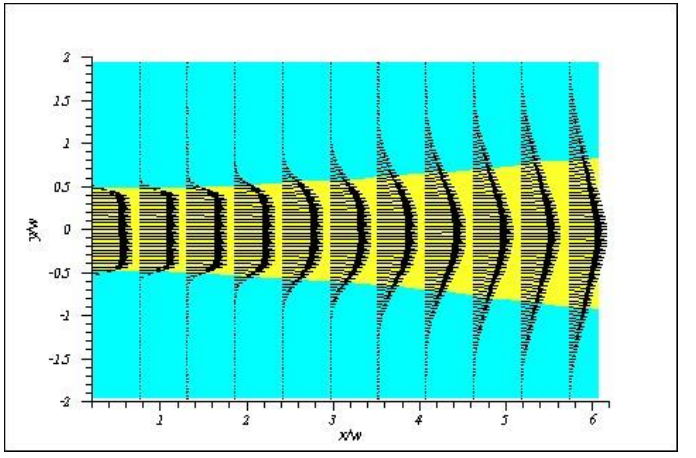


Ludwig Prandtl
1875 - 1929

In the two-dimensional flow the value of the mixing length is determined by empirical formulae, for example:

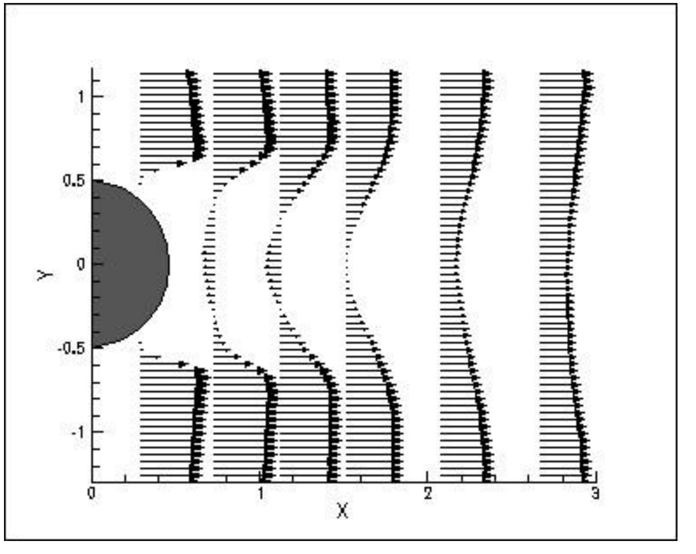
An ouflow: $l_m = 0{,}09 \cdot L$

L – half width of the stream



A wake: $l_m = 0{,}16 \cdot L$

L – half width of the wake



Channel of width 2L or a pipe of radius L

$$l_m = L \cdot \left[0{,}14 - 0{,}08 \cdot \left(1 - \frac{y}{L}\right)^2 - 0{,}06 \cdot \left(1 - \frac{y}{L}\right)^4 \right]$$

The zero-equation model may be developed for three-dimensional flows

Advantages of the zero-equation model:

-Easily applicable and „cheap’ in the sense of low computation cost

-Produces good results for thin shearing layers, outflows from orifices and wakes behind solid objects

-Well established – large experience in its application has been accumulated

Disadvantages of the zero-equation model:

-Does not take into account the „history” of the flow

-Does not take into account the kinetic energy of turbulence

-Fails in the cases of flows with separation and recirculation

One-equation model

This model relates the kinematic coefficient of turbulent viscosity to the konetic energy of turbulence *k*. This energy is determined by means of an additional equation, which must be solved numerically together wityh the Reynolds equation and mass conservation equation.

$$\nu_t = l_m \cdot k^{\frac{1}{2}}$$

Turbulent viscosity coefficient

$$\frac{\partial(\rho k)}{\partial t} + \sum_i U_i \frac{\partial k}{\partial x_i} = \sum_i \frac{\partial}{\partial x_i}\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_i} +$$

$$+ \sum_{ij} \tau_{ij} \frac{\partial U_i}{\partial x_j} - C_{k2} \frac{k^{\frac{3}{2}}}{l_m}$$

Equation describing the variation of the turbulent kinetic energy in the flow domain

Physical interpretation of the equation for the turbulent kinetic energy is as follows:

$$A+B=C+D+E$$

A – local variation of *k*

B – convective variation of *k*

C – transport of *k* through diffusion

D – „production” of *k* through rate of strain

E – dissipation of the turbulent kinetic energy

The quantities l, C_{k2}, σ_k are empirically determined constants

In comparison with the zero-equation model, the one-equation model enables taking into account the history of variation of the turbulent kinetic energy in the flow.

The two-equation model (1974)

The two-equation model introduces two additional equations: for turbulent kinetic energy *k* and for the velocity of its dissipation ε. These equations must be solved together with the Reynolds equations and the mass conservation equation. These two equations may be developed theoretically from the Navier-Stokes equation and Reynolds equation, but they require additional empirical coefficients. In the standard *k*-ε model these equations have the following form:

$$\frac{\partial(\rho k)}{\partial t} + div\left(\rho k \overline{U}\right) = div\left(\frac{\mu_t}{\sigma_k} grad k\right) + 2\mu_t E_{ij} E_{ij} \rho \varepsilon$$

$$\frac{\partial(\rho \varepsilon)}{\partial t} + div\left(\rho \varepsilon \overline{U}\right) = div\left(\frac{\mu_t}{\sigma_\varepsilon} grad \varepsilon\right) + C_{1\varepsilon} \frac{\varepsilon}{k} 2\mu_t E_{ij} E_{ij} - C_{2\varepsilon} \rho \frac{\varepsilon^2}{k}$$

Where: $$E_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

Physical interpretation of both equations is similar: A+B=C+D+E

A – local variation of k or ε

B – convective variation of k or ε

C – transport of k or ε through diffusion

D – production of k or ε through rate of strain

E – dissipation of k or ε

The following coefficients are empirically determined:

$$C_\mu = 0{,}09 \qquad \sigma_k = 1{,}0 \qquad \sigma_\varepsilon = 1{,}3 \quad C_{1\varepsilon} = 1{,}44 \qquad C_{2\varepsilon} = 1{,}92$$

$$\mu_t = \rho \cdot C_\mu \frac{k^2}{\varepsilon}$$ Dynamic turbulent viscosity coefficient

Variation of the above is the k-ω model, where: $\omega \approx \frac{k}{\varepsilon}$

Advantages of the two-equation model:

-The simplest „true” turbulence model

-Produces good results for many realistic, technologically meaningful flows

-One of the best verified models

Disadvantages of the two-equation models:

-More „expensive” than the mixing length model

-Poor results for several practically important flows, such as: vorticity-dominated flows, flows with very high shearing stresses etc.

Some of the disadvantages of the two-equation model (first of all its isotropy), may be eliminated by direct modelling of the Reynolds stressest (RSM – Reynolds Stress Modelling), which **requires seven additional equations.**

Seven equation model - RSM

In this model the Reynolds stresses are modelled directly by 6 equations of the following form:

$$\frac{DR_{ij}}{Dt} = P_{ij} + D_{ij} - \varepsilon_{ij} + \Pi_{ij} + \Omega_{ij}$$

$\frac{DR_{ij}}{Dt}$ - local and convective variation of the Reynolds stresses (material derivative)

P_{ij} - „production" of stresses

D_{ij} - „transport" of stresses through diffusion

ε_{ij} - dissipation of stresses

Π_{ij} - „transport" of stresses through interaction of the pressure field with rates of strain

Ω_{ij} - „transport" of stresses through vortex motion of the fluid

The seventh equation of the RSM model is the relation for dissipation of the turbulent kinetic energy, identical as in the *k*-*ε* model:

$$\frac{\partial(\rho\varepsilon)}{\partial t}+div\left(\rho\varepsilon\overline{U}\right)=div\left(\frac{\mu_t}{\sigma_\varepsilon}grad\varepsilon\right)+C_{1\varepsilon}\frac{\varepsilon}{k}2\mu_t E_{ij}E_{ij}-C_{2\varepsilon}\rho\frac{\varepsilon^2}{k}$$

Additionally, the following relation for determination of the turbulent kinetic energy is used:

$$k=\frac{1}{2}\left(R_{11}+R_{22}+R_{33}\right)=\frac{1}{2}\left(u_1'^2+u_2'^2+u_3'^2\right)$$

Advantages of the RSM model

Potentially the most general of the classical turbulence models

Requires only boundary and intial conditions

Delivers very accurate values of the mean velocity field and of Reynolds stresses for many complicated and simple flows

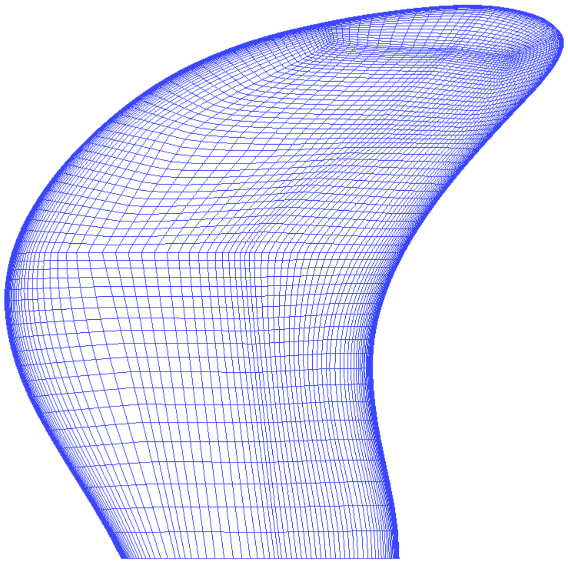
Disadvantages of the RSM model

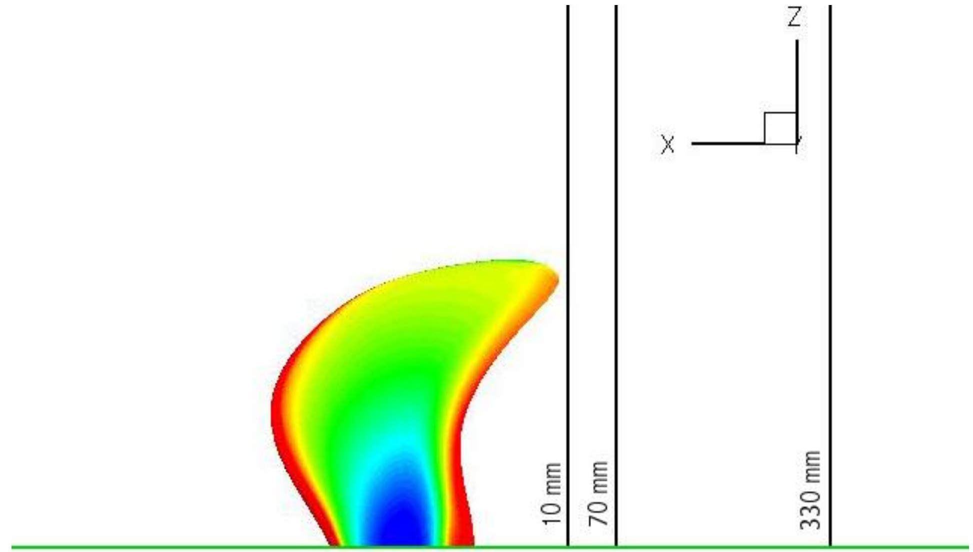
Requires very high computation time – 7 additional equations

It is not as thoroughly verified as the simpler turbulence models

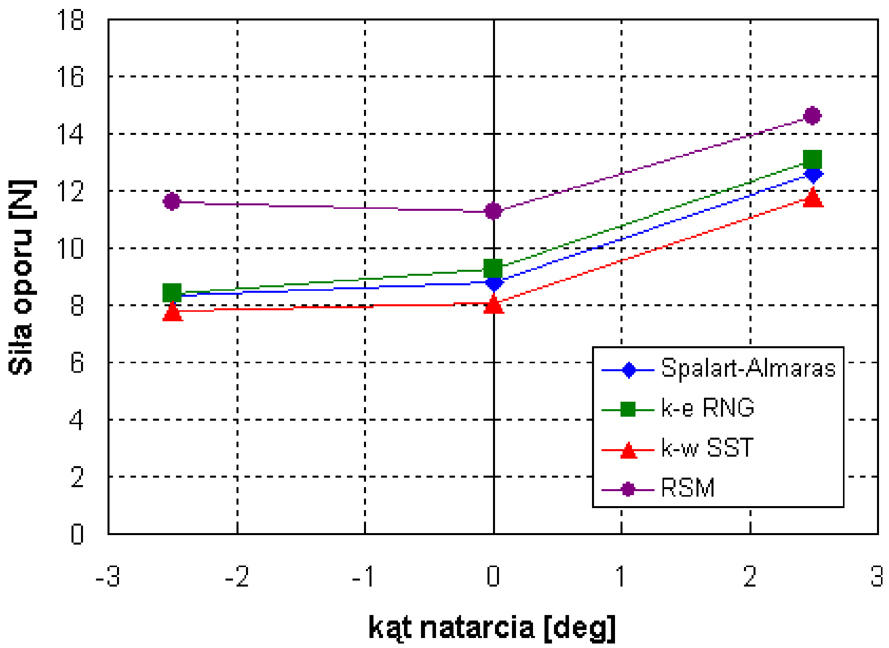
In certain applications it works as poorly as other, simpler models

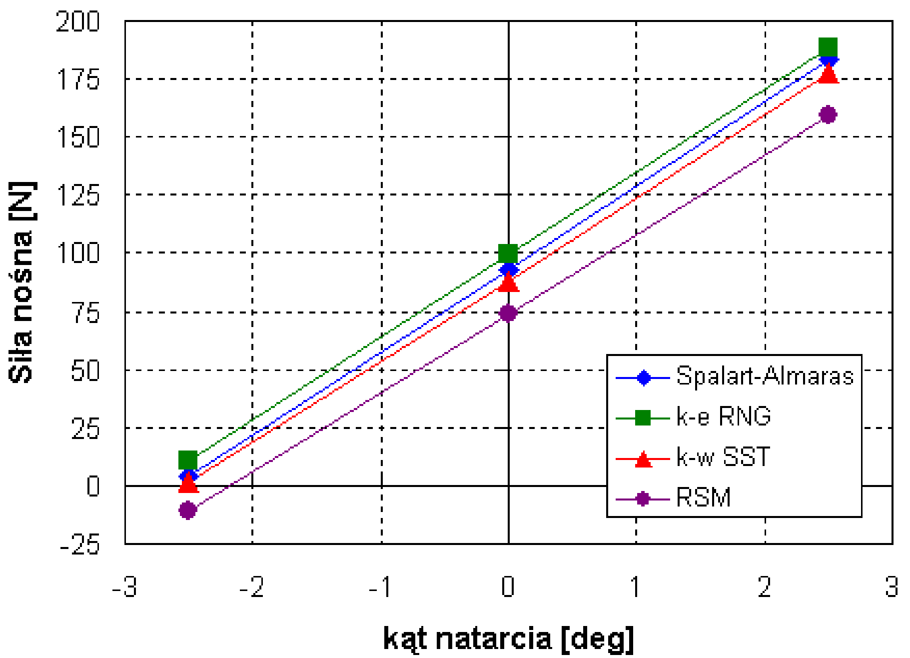
Example of application of different turbulence models to a practical computational problem





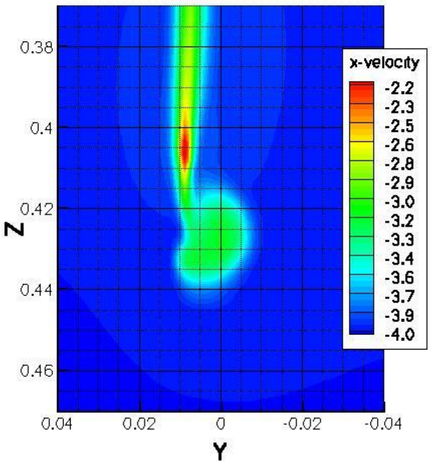
The problem requires determination of the velocity field in the wake behind a lifting foil, in three cross-sections located at 10, 70 and 330 mm behind the trailing edge. Four turbulence models are applied: one-equation (Spalart-Allmaras), two-equation k-ε RNG and k-ω SST and seven-equation RSM (Reynolds Stress Modelling)



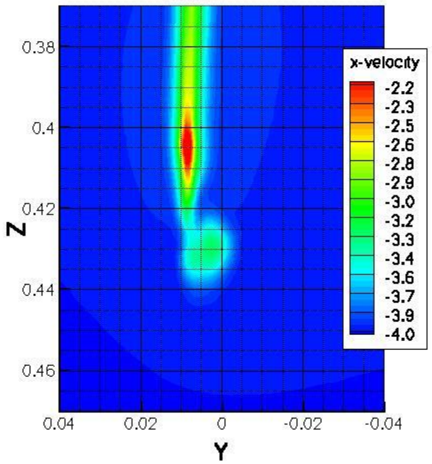


Dependence of the hydrofoil drag force and lift force on the angle of attack of the hydrofoil, calculated using different turbulence models

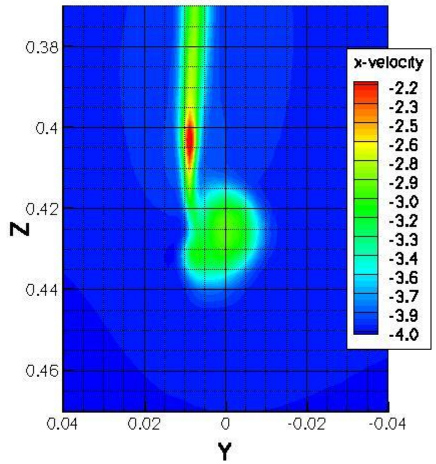
Axial velocity component in cross-section 10 mm behind the foil



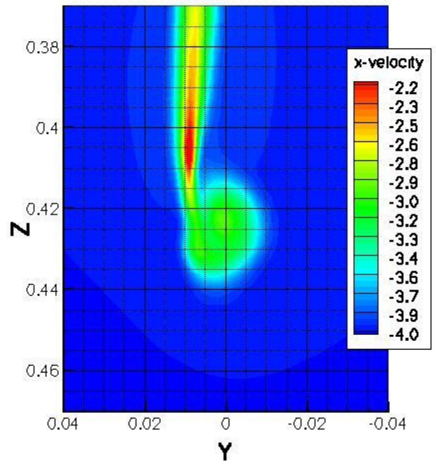
Spalart



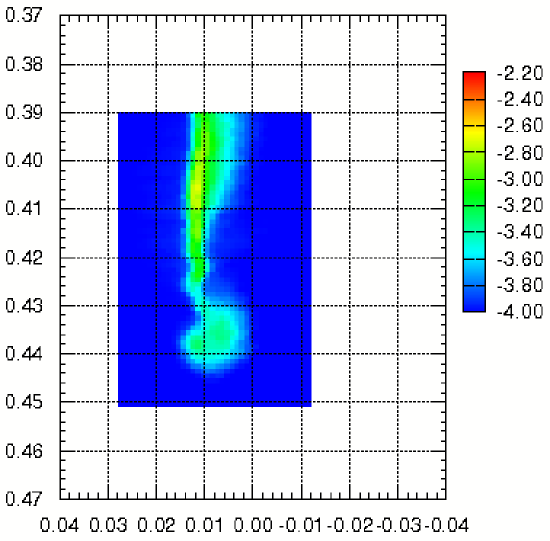
k-epsilon



k-omega

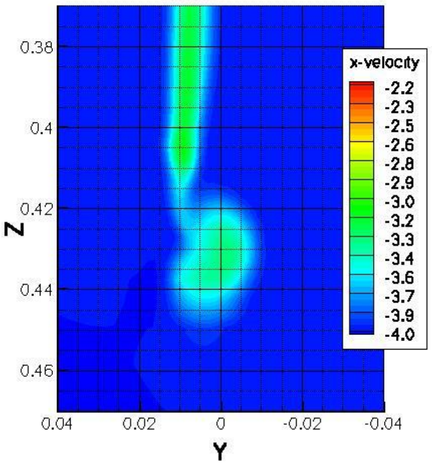


RSM

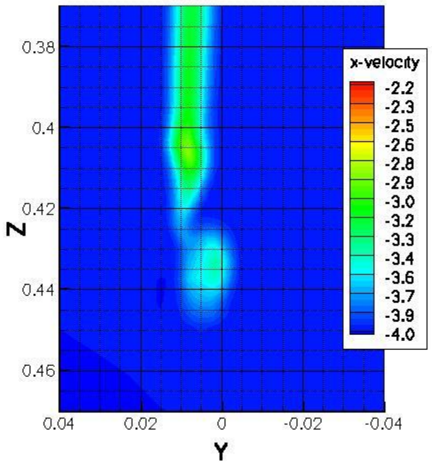


<- Results of LDV measurements

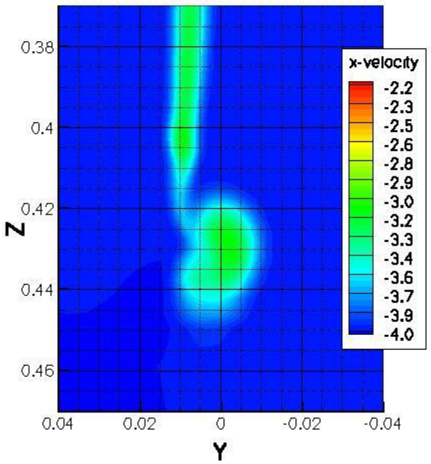
Axial velocity component in cross-section 70 mm behind the foil



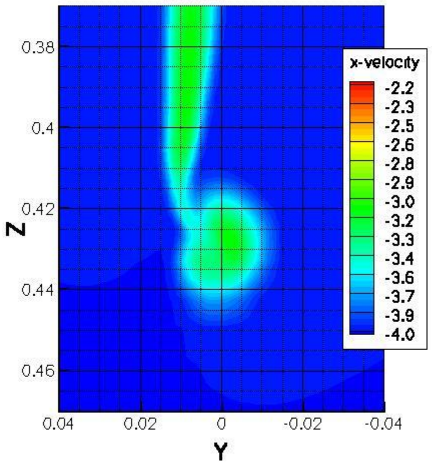
Spalart



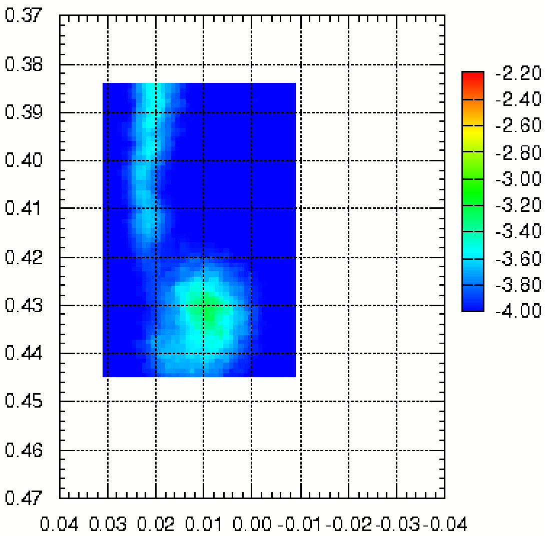
k-epsilon



k-omega

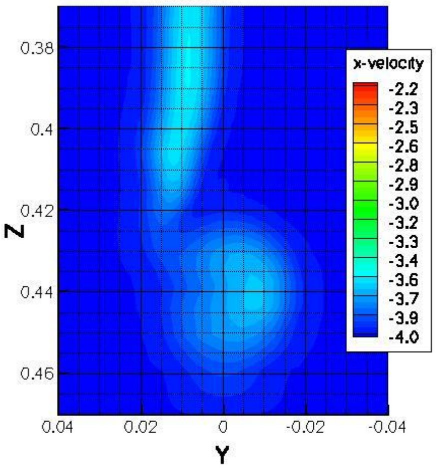


RSM

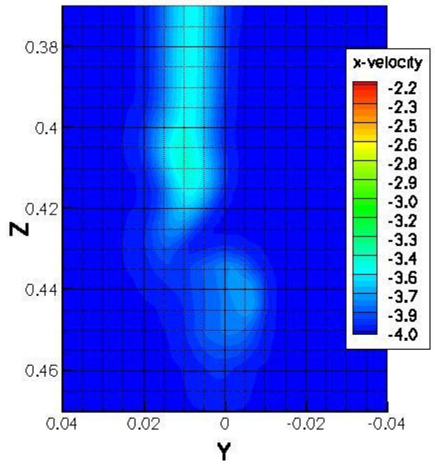


<-Results of LDV measurements

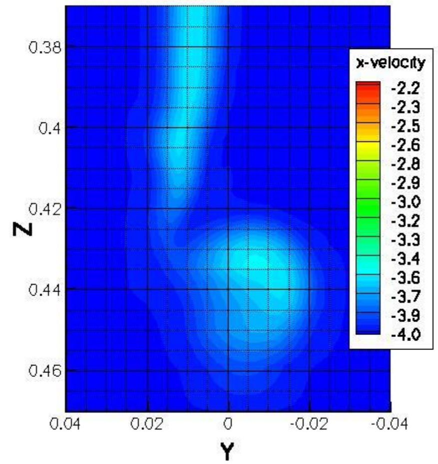
Axial velocity component in cross-section 330 mm behind the foil



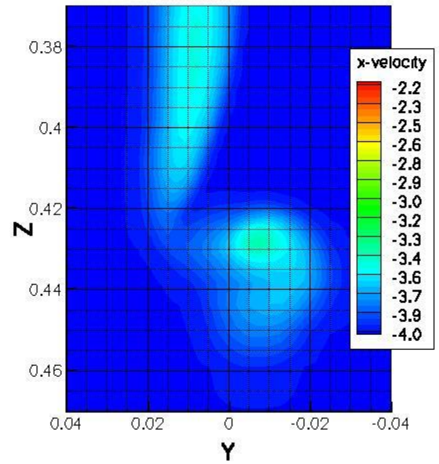
Spalart



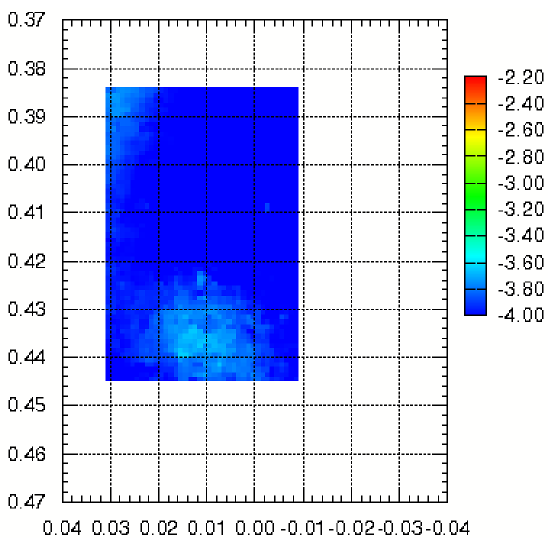
k-epsilon



k-omega



RSM



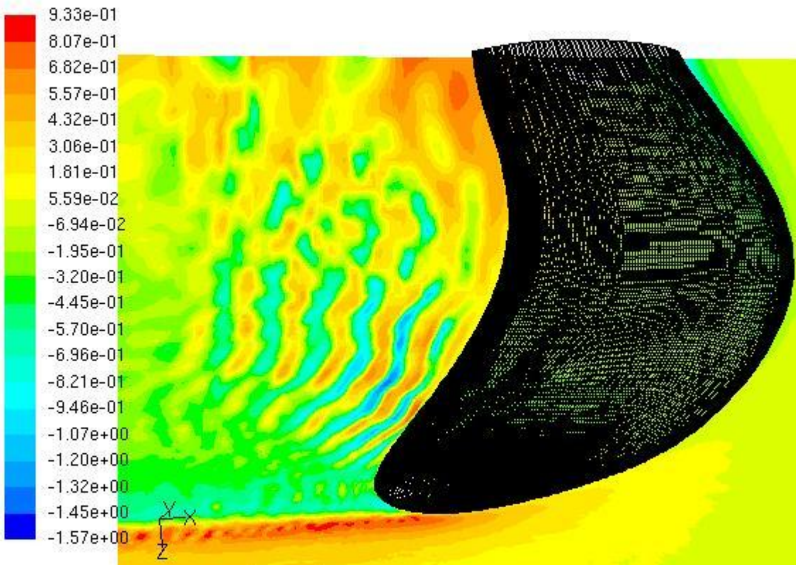
<-Results of LDV measurements

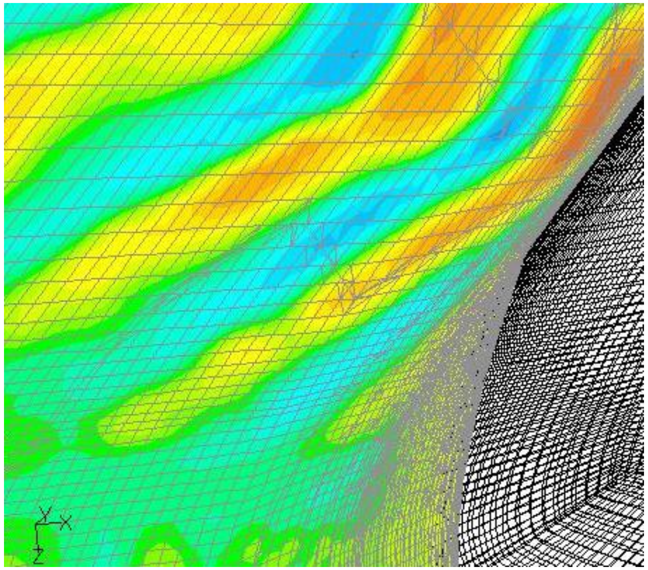
An alternative to the application of Reynolds equations together with the turbulence models (or RANSE method) is the so called LES (Large Eddy Simulation method) or DNS (Direct Numerical Simulation method).

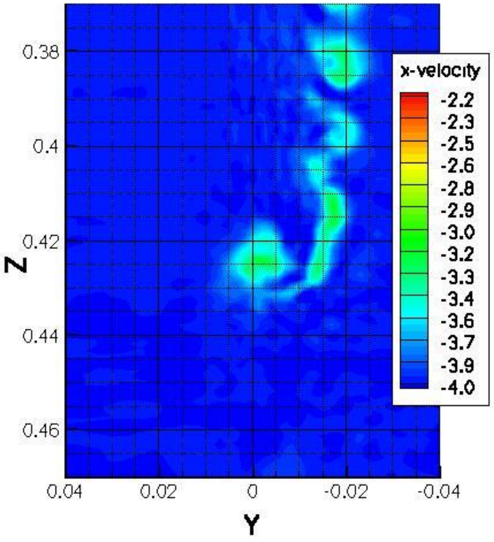
LES method is based on numerical simulation of large, coherent vortex structures and on modelling of the small turbulent vortices (below the size of the computational grid) by selected model equations. It requires large computer memory and long computation time.

DNS method is based on direct numerical simulation of the entire Kolmogorov turbulence cascade of vortices down to the smallest turbulence scales. Its application requires very large computing resources and nowadays it is not used for solution of practical engineering flows yet.

Flow behind a foil calculated using LES method – the numerically simulated large, coherent vortex structures are visible



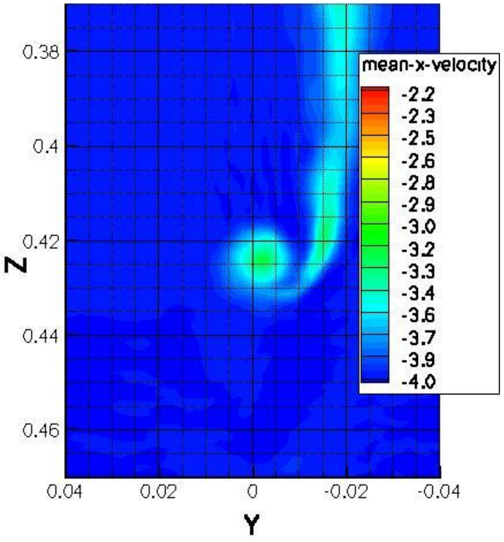




Axial velocity in section 70 mm behind the foil

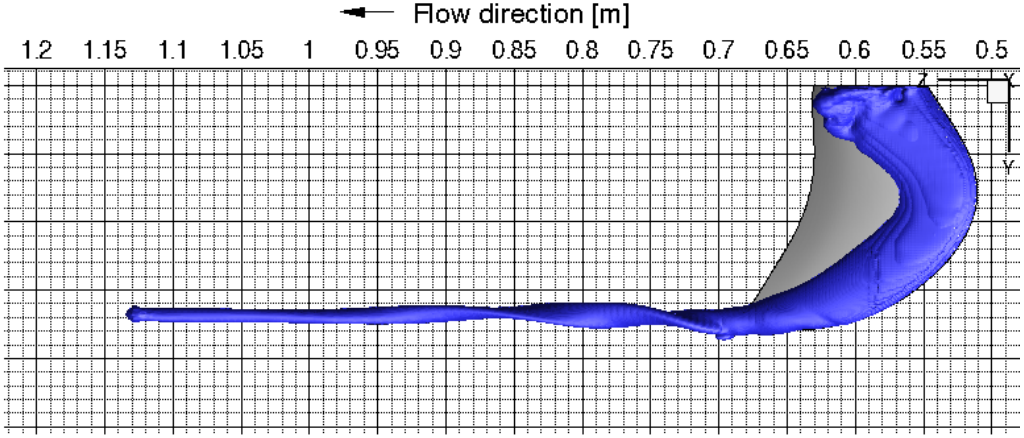
<- transient values

mean values->



Calculations of the velocity field behind a foil are only an introduction to the numerical prediction of vortex cavitation

Calculation (Fluent)



Experiment in the cavitation tunnel