# J. Szantyr – Lecture No. 6 – Fluid – Solid Interaction – Concept of the Entrained Mass of Fluid

In 1828 Friedrich Bessel has noticed that a pendulum immersed in water changes (increases) its period of oscillations in comparison to the value in air. This may be interpreted as a virtual increase of mass of the pendulum. Bessel has introduced the idea of the **entrained mass of water**, i.e. a certain mass of water (in general: fluid), exercising the unsteady motion **together with the immersed object** and changing its motion characteristics. The entrained mass increases the inertia of the object, introducing additional forces to the description of motion.

Friedrich Wilhelm Bessel
1784 - 1846

Alfred Barnard Basset
1854 - 1930

The additional forces on the object exercising an accelerated motion in a real, viscous fluid (in contrast to the motion in a vacuum) may be divided into two parts: a part associated with acceleration of a certain mass of fluid (in principle a completely potential effect) and a part resulting from viscous effects in the unsteady boundary layer forming on the solid object. This second part is called **Basset force (1888)**.

**The Basset force** is important first of all for small solid objects moving in a fluid. Its magnitude depends on the history of motion and in the case of a spherical object it may be described by the following formula:

$$\overline{F}(t) = \frac{3}{2} D^2 \sqrt{\pi \rho_C \mu_C} \int_0^t \frac{\frac{D\overline{u}}{Dt'} - \frac{D\overline{v}}{Dt'}}{\sqrt{t-t'}} dt'$$

where:

$D$ – object diameter

$t$ – current time

$\rho_C$ - fluid density

$\mu_C$ - fluid dynamic viscosity coefficient

$\overline{u}$ - velocity of the object

$\overline{v}$ - velocity of the fluid

In the physical sense the Basset force results from a retarded formation of the boundary layer and viscous wake behind the solid object moving in the fluid in an accelerated way.

The simplest interpretation: **entrained mass** determines the work required to change the kinetic energy of the fluid due to an accelerated motion of the immersed solid object. The kinetic energy of the fluid motion caused by the moving solid object may be written as:

$$E = \frac{\rho}{2} \int_V \left(u_1^2 + u_2^2 + u_3^2\right) dV$$

where $V$ – the entire fluid volume

In a steady linear motion there is $E=const$ and $E \propto U^2$

Then we may write: $E = \frac{\rho}{2} I U^2$ where: $I = \int_V \left[ \left(\frac{u_1}{U}\right)^2 + \left(\frac{u_2}{U}\right)^2 + \left(\frac{u_3}{U}\right)^2 \right] dV$

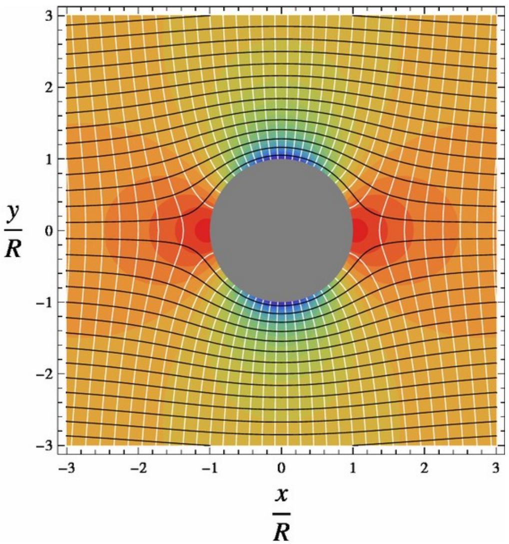
If the object accelerates or brakes then the energy $E$ changes with the velocity $U$. The change of energy $E$ may be caused only by the work of an additional hydrodynamic force $F$, which appears on the object in an unsteady motion, according to the relation:

$$F = -\frac{1}{U}\frac{dE}{dt} = -\rho I \frac{dU}{dt}$$

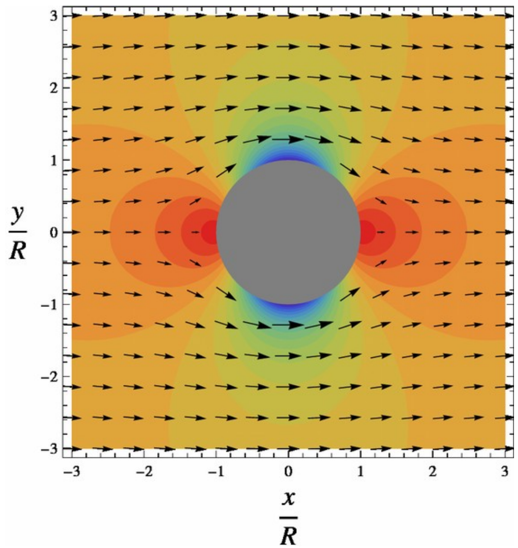
force $F$ is similar to the force required to accelerate the object of mass $m$ *i.e.*: $m\, dU/dt$

It is convenient to describe the force $F$ as an additional mass of fluid $M=\rho I$ accelerated together with the object. In reality **every fluid particle around the object experiences different acceleration**, hence the entrained mass $M$ is a certain „virtual mass.”.

# A simple example – linear accelerated motion of a sphere or a cylinder in a two-dimensional flow (2D):

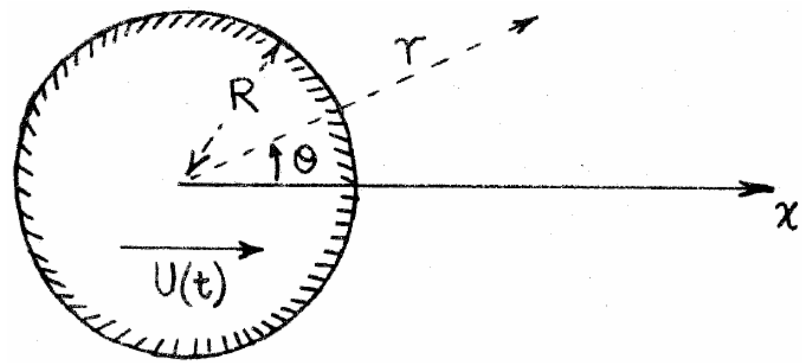



Stream lines and equipotential lines




Velocity vectors and pressure field

The potential fluid flow description is applied




The following velocity potentials are obtained:

For the sphere: $$\Phi(r,\vartheta) = -\frac{UR^3}{2r^2}\cos\vartheta$$

For the cylinder: $$\Phi(r,\vartheta) = -\frac{UR^2}{r}\cos\vartheta$$

Then the integrals $I$ determining the entrained mass may be calculated as:

For the sphere:

$$I = \int_R^\infty \int_0^{2\pi} \left[ \left( \frac{1}{U} \frac{\partial \Phi}{\partial r} \right)^2 + \left( \frac{1}{Ur} \frac{\partial \Phi}{\partial \vartheta} \right)^2 \right] 2\pi r^2 \sin\vartheta \, d\vartheta \, dr = \frac{2}{3} \pi R^3$$

i.e. it is equal to half of the fluid mass displaced by the sphere

For the cylinder (per unit length):

$$I = \int_R^\infty \int_0^{2\pi} \left[ \left( \frac{1}{U} \frac{\partial \Phi}{\partial r} \right)^2 + \left( \frac{1}{Ur} \frac{\partial \Phi}{\partial \vartheta} \right)^2 \right] r \, d\vartheta \, dr = \pi R^2$$

i.e. it is equal to the fluid mass displaced by the cylinder

In the general case of motion of an object in six degrees of freedom the unsteadiness of any velocity component results in generation of additional forces in all six degrees of freedom. Then we obtain a matrix (a tensor) of entrained masses: $M_{ij}$

$$F_i = -M_{ij} \dot{u}_j \qquad i,j=1,2,3,4,5,6$$

It may be shown that in a potential flow the matrix of entrained masses is symmetrical, hence in a general case we may have 21 independent entrained masses. Symmetry of the moving solid object may lead to further reduction of the number of entrained masses.

# Tensor of the entrained masses

Symmetry:

$M_{ij} = M_{ji}$

Dimensions

*kg*

*kgm*

$kgm^2$

$$
F = \begin{bmatrix}
M_{11} & M_{12} & M_{13} & M_{14} & M_{15} & M_{16} \\
M_{21} & M_{22} & M_{13} & M_{24} & M_{25} & M_{26} \\
M_{31} & M_{32} & M_{33} & M_{34} & M_{35} & M_{36} \\
M_{41} & M_{42} & M_{43} & M_{44} & M_{45} & M_{46} \\
M_{51} & M_{52} & M_{53} & M_{54} & M_{55} & M_{56} \\
M_{61} & M_{62} & M_{63} & M_{64} & M_{65} & M_{66}
\end{bmatrix}
\begin{bmatrix}
\dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \dot{u}_4 \\ \dot{u}_5 \\ \dot{u}_6
\end{bmatrix}
$$

First index – direction of force, second index – direction of motion

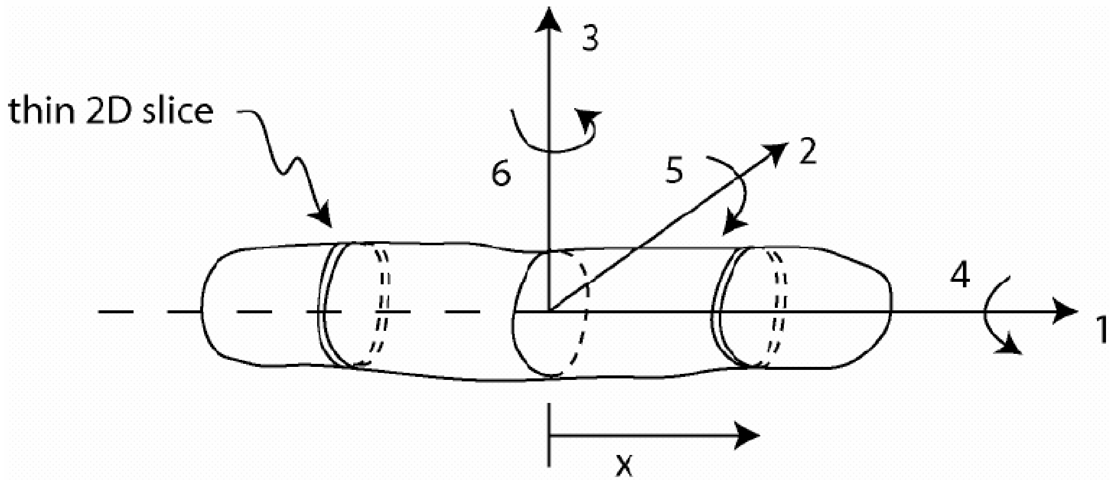
i=1,2,3 – forces; i=4,5,6 - moments

j=1,2,3 – linear accelerations; j=4,5,6 – angular accelerations

Sometimes the entrained masses are presented in a non-dimensional form, i.e. related to the respective mass characteristic of the solid object.

Non-dimensional coefficient of the entrained masses are denote $m_{ij}$

Calculation of the entrained mass coefficients for a three-dimensional object of an arbitrary geometry is difficult. If one dimension of the object is significantly larger than others, then the so called slender body theory may be applied.




In this theory the object may be cut into „slices" and the entrained mass coefficients for two-dimensional sections may be integrated along the object:

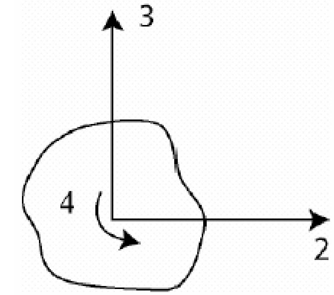
$$m_{22} = \int_L a_{22} dx \qquad m_{23} = -\int_L a_{23} dx \qquad m_{24} = \int_L a_{24} dx \qquad m_{26} = \int_l x a_{22} dx$$

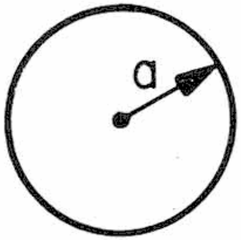
$$m_{33} = \int_L a_{33} dx \qquad m_{35} = -\int_L x a_{33} dx \qquad m_{44} = \int_L a_{44} dx \qquad m_{46} = \int_L x a_{24} dx$$

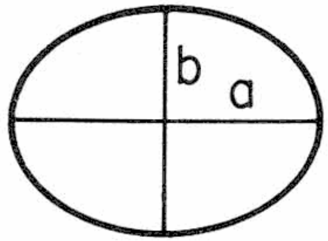
$$m_{55} = \int_L x^2 a_{33} dx \qquad m_{66} = \int_L x^2 a_{22} dx$$

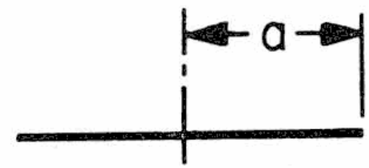
In more complicated cases the commercial CFD software is used

The entrained mass coefficients for some selected two-dimensional sections are given below:

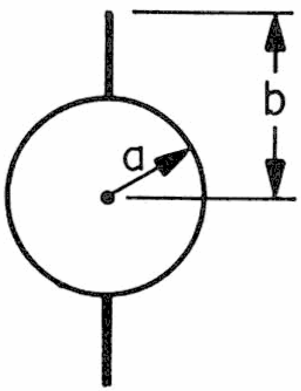


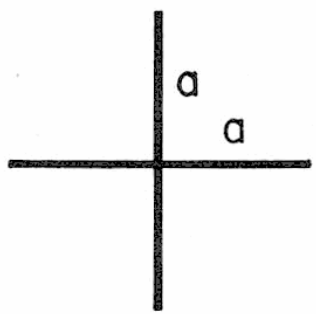



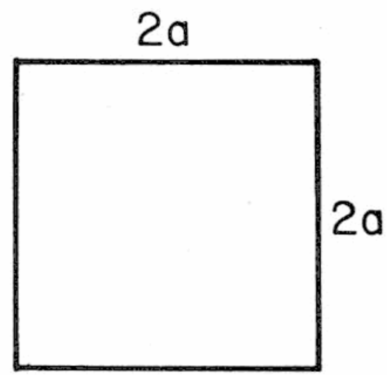




| | | | |
|---|---|---|---|
| $m_{11}$: | $\pi\rho a^2$ | $\pi\rho b^2$ | $0$ |
| $m_{22}$: | $\pi\rho a^2$ | $\pi\rho a^2$ | $\pi\rho a^2$ |
| $m_{66}$: | $0$ | $\frac{1}{8}\pi\rho(a^2 - b^2)^2$ | $\frac{1}{8}\pi\rho a^4$ |



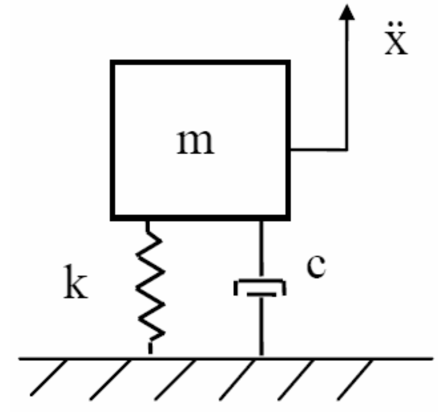

| | | | |
|---|---|---|---|
| $m_{11}$: | $\pi\rho[a^2 + (b^2 - a^2)^2/b^2]$ | $\pi\rho a^2$ | $4.754\ \rho a^2$ |
| $m_{22}$: | $\pi\rho a^2$ | $\pi\rho a^2$ | $4.754\ \rho a^2$ |
| $m_{66}$: | * | $\frac{2}{\pi}\rho a^4$ | $0.725\ \rho a^4$ |

In an unsteady motion of the solid object immersed in fluid **the entrained mass** is a virtual mass of fluid performing motion with the same velocity as the solid object. The entrained mass increases the inertia of the object and in this way it influences the motion characteristics of the object.

In reality the motion of the immersed solid object induces the motion of **another** mass of fluid with diverse velocities – higher velocity close to the object and smaller at larger distances from it. This **real** mass of moving fluid increases the inertia of the object **in the same way as the virtual entrained mass**.

For objects moving in gases the entrained mass of gas is usually not taken into account due to the small density of gases.

**Influence of the entrained mass on the solid object oscillations – a simple one-dimensional example.**




$m$ – mass of the object

$c$ – damping coefficient (due to fluid viscosity)

$k$ – restoring force coefficient

$x$ – object displacement

The entrained mass increases the inertia of the object, thus it counteracts oscillations. In this case the equation describing oscillations has the form:

$$m\ddot{x} + c\dot{x} + kx = -m_a\ddot{x}$$

$$\left(m + m_a\right)\ddot{x} + c\dot{x} + kx = 0$$

$$m_e\ddot{x} + c\dot{x} + kx = 0$$

where:

$m_a$ - entrained mass

$m_e$ - „effective" mass

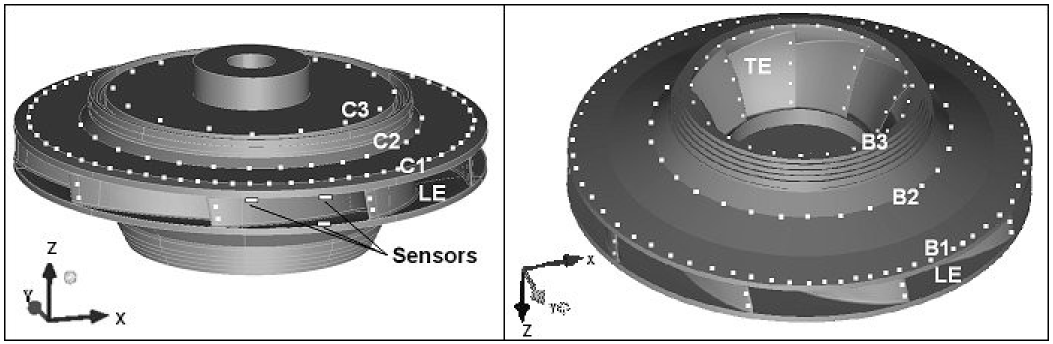
The own frequency of oscillations of the immersed body may be determined as:

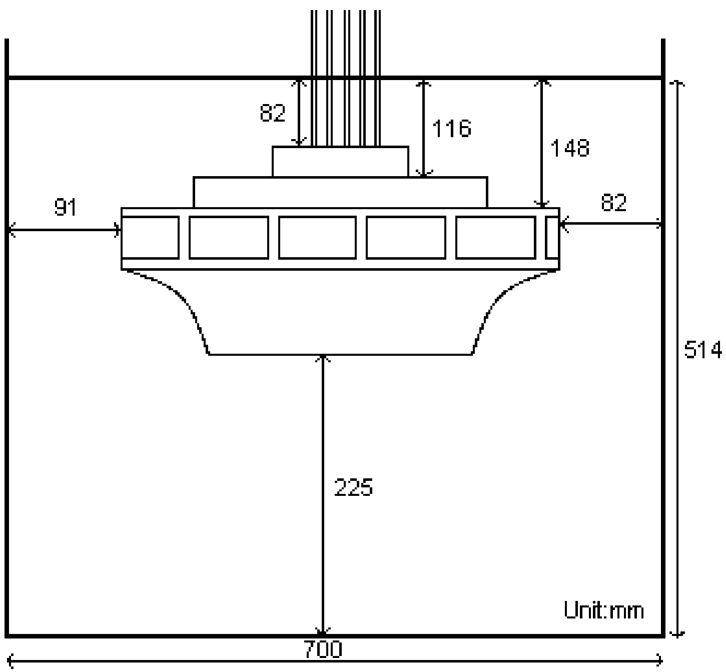
$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m_e}}\sqrt{1 - \frac{c^2}{4m_e k}}$$

It should be noticed that immersion of the oscillating object results in reduction of the own frequency of oscillations.

# Influence of the entrained mass on the vibration of the reversible machine (pump-turbine) rotor

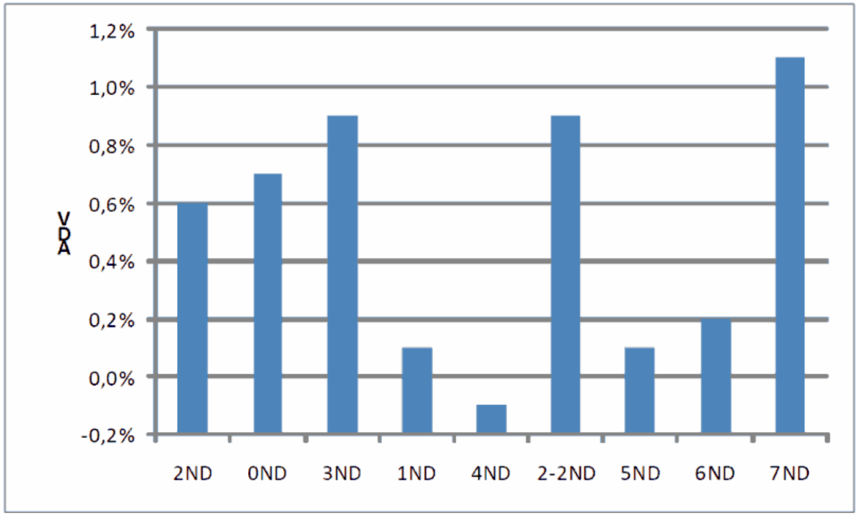
## Model experiments



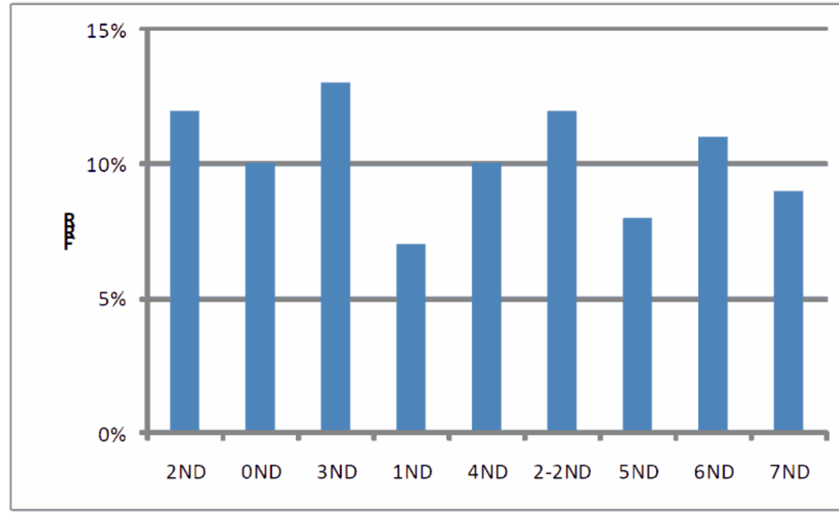




Model of the rotor of a reversible machine (pump-turbine) has been tested in air and in water. Model vibration was excited by an inducer in 384 points shown in the picture.  The responses for several own modes of vibration were registered.

Change of the damping coefficients




Change of own frequencies




Own frequencies and damping coefficients in air and in water

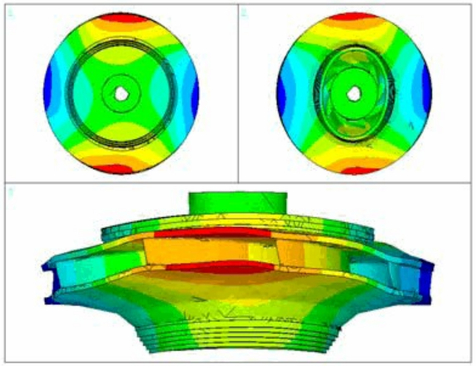
**Table 1** Averaged natural frequencies, damping ratios, FRR's and ADV's in air and in water.

| Mode (ND) | $f_a$ | $f_w$ | $FRR$ | $\zeta_a$ | $\zeta_w$ | $ADV$ |
|---|---|---|---|---|---|---|
| 2ND | 838 | 735 | 0,12 | 0,034 | 0,040 | 0,006 |
| 0ND | 1111 | 1006 | 0,10 | 0,030 | 0,037 | 0,007 |
| 3ND | 1463 | 1279 | 0,13 | 0,032 | 0,041 | 0,009 |
| 1ND | 1439 | 1342 | 0,07 | 0,030 | 0,031 | 0,001 |
| 4ND | 1811 | 1630 | 0,10 | 0,032 | 0,031 | -0,001 |
| 2-2ND | 1861 | 1630 | 0,12 | 0,022 | 0,031 | 0,009 |
| 5ND | 2282 | 2103 | 0,08 | 0,025 | 0,026 | 0,001 |
| 6ND | 2769 | 2464 | 0,11 | 0,014 | 0,016 | 0,002 |
| 7ND | 3003 | 2741 | 0,09 | 0,015 | 0,026 | 0,011 |

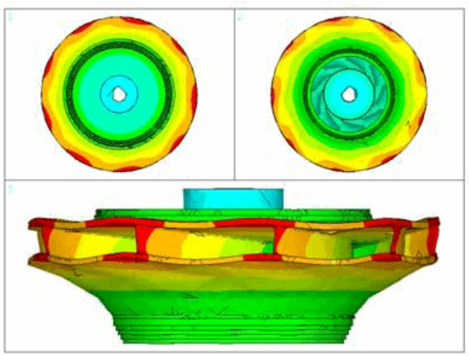
# Influence of the entrained mass on the vibration of the pump rotor

## Numerical calculations

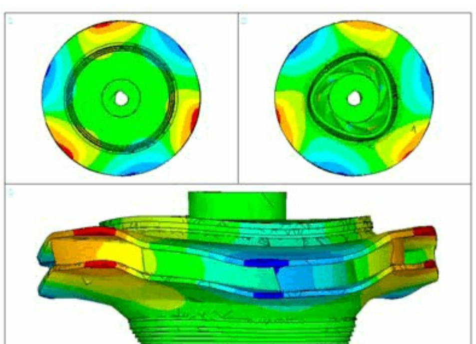
The basic own vibration modes



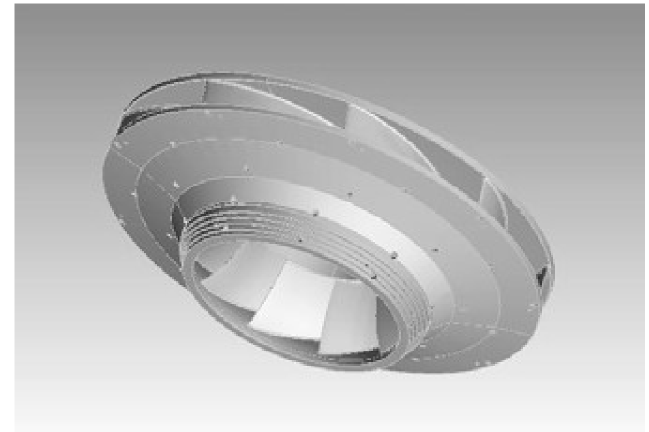
2ND



0ND

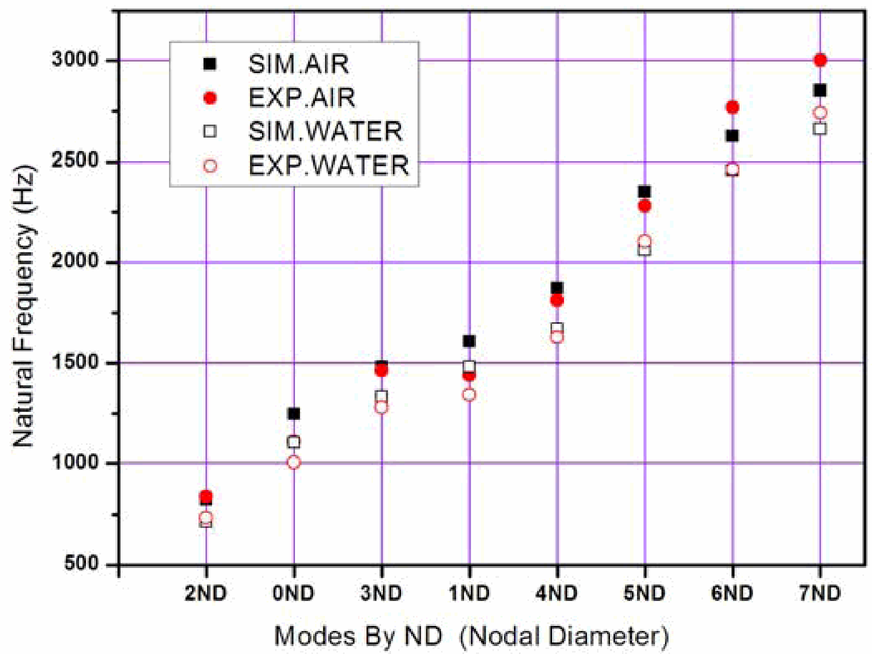


3ND



Model of the rotor

Calculations were performed using the Finite Element Method. The computational model of the rotor was built of 165000 quadrihedral elements, and the model of the surrounding fluid was built of 342676 such elements.

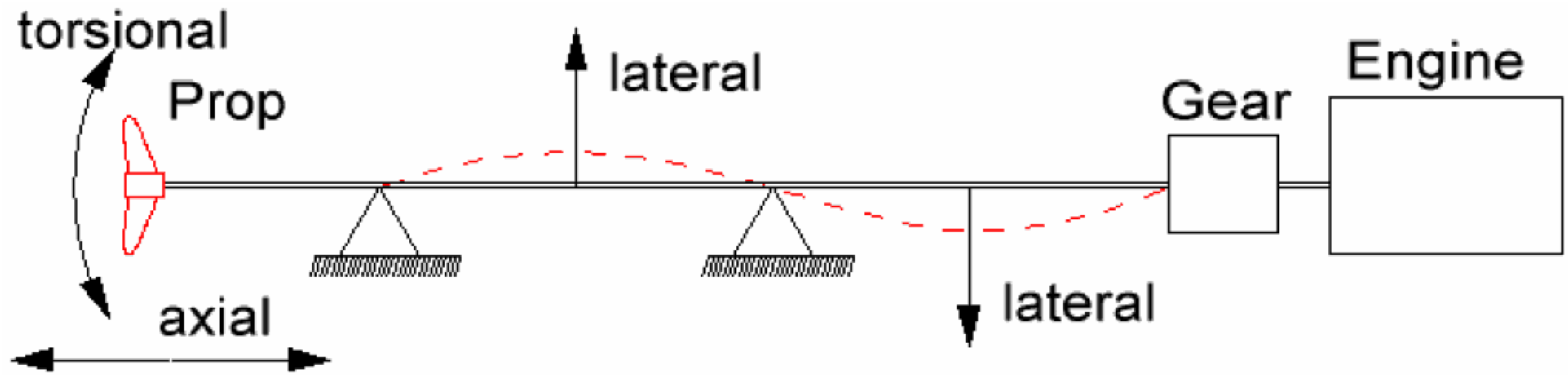



Comparison of the calculated (SIM) and measured (EXP) own vibration frequencies of the rotor in air and in water

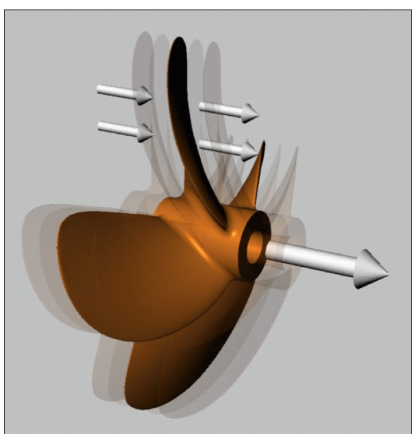
| | SIM.AIR | EXP.AIR | SIM.WATER | EXP.WATER | SIM.RATIO | EXP.RATIO |
|---|---|---|---|---|---|---|
| **2ND** | 825.26 | 838.00 | 714.11 | 734.50 | 0.87 | 0.88 |
| **0ND** | 1247.40 | 1111.00 | 1103.80 | 1006.00 | 0.88 | 0.90 |
| **3ND** | 1479.75 | 1463.00 | 1332.90 | 1279.00 | 0.90 | 0.87 |
| **1ND** | 1605.55 | 1439.00 | 1482.05 | 1342.00 | 0.92 | 0.93 |
| **4ND** | 1871.80 | 1811.00 | 1668.90 | 1629.00 | 0.89 | 0.90 |
| **5ND** | 2350.15 | 2282.00 | 2062.05 | 2103.60 | 0.88 | 0.92 |
| **6ND** | 2624.25 | 2769.00 | 2458.80 | 2464.00 | 0.94 | 0.89 |
| **7ND** | 2852.00 | 3002.50 | 2661.15 | 2741.50 | 0.93 | 0.91 |

**Table 3.** Results of modal analysis of pump-turbine runner

# Influence of the entrained mass on the vibration of the ship propulsion system

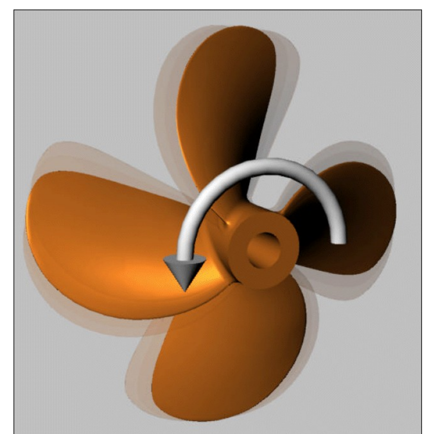



Scheme of a ship  propulsion system. The most important are the torsional and longitudinal (axial) vibrations. An important component of the system is the propeller, being a heavy object immersed in water. Variable hydrodynamic forces are generated on the propeller, constituting the main source of vibration excitation.



←Variable thrust force

Variable torque on the shaft -->



Determination of the entrained mass for the propeller is necessary for correct analysis of vibration of the ship propulsion system. There are many methods for determination of the entrained mass. The simplest are the empirical formulae:

For axial vibration:

$$M_{11} = (0{,}1 - 0{,}2) M_P$$

or:

$$M_{11} = C_{11} \rho D^3$$

For torsional vibration:

$$M_{44} = K_1 \frac{M_P D}{K_2}$$

or:

$$M_{44} = C_{44} \rho D^5$$

$M_P$ - mass of the propeller

$D$ – propeller diameter

$$K_1 = 0{,}25 - 0{,}30$$

$$K_2 = 19 - 28$$

$$C_{11} = \frac{0{,}2 \left( P/D \right)^2 A^2}{z}$$

$$C_{44} = \frac{0{,}0224 \left( P/D \right)^2 A^2}{z}$$

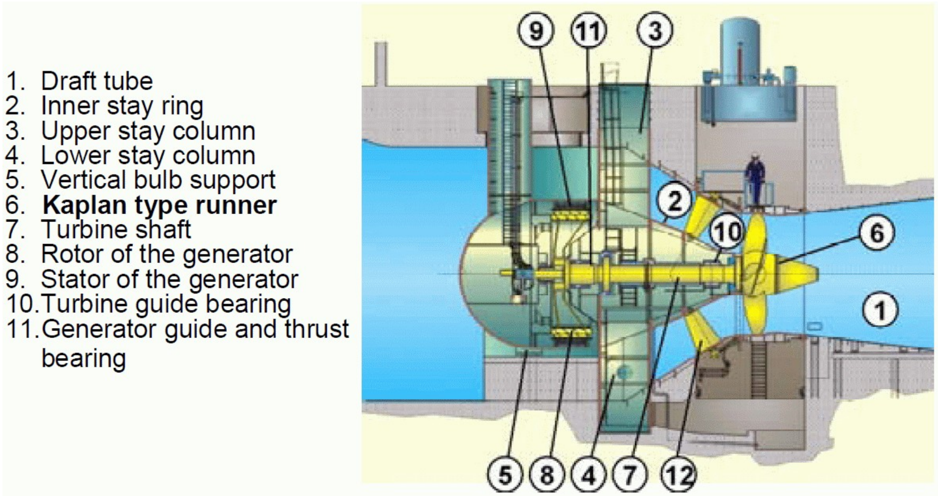
$P$ – propeller pitch

$z$ – propeller number of blades

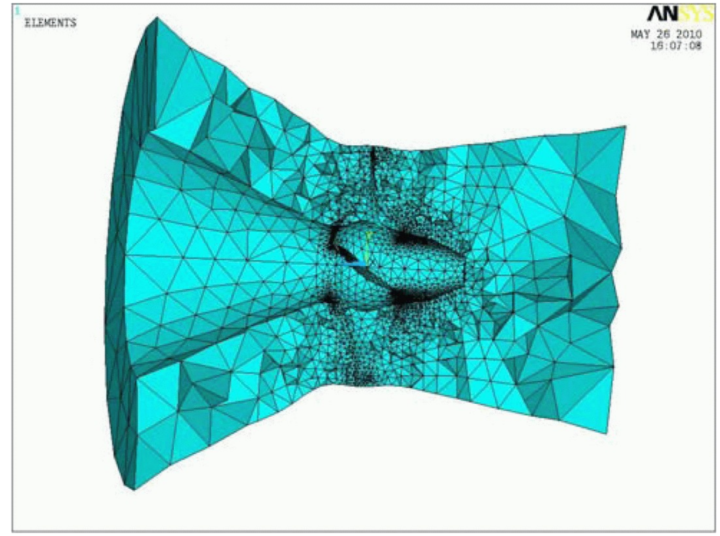
$A$ – propeller area coefficient

$\rho$ - density of water

# Computational determination of the entrained mass for a Kaplan turbine



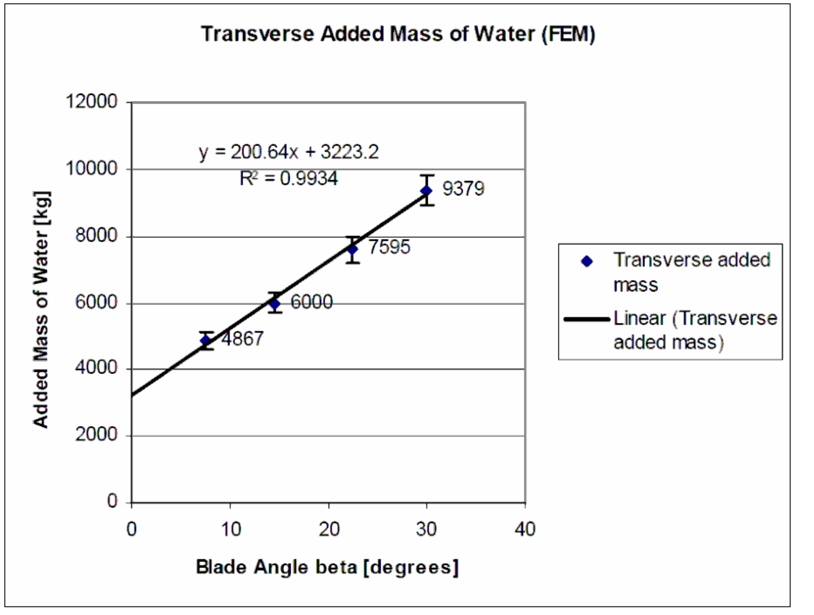

Scheme of the Kaplan turbine

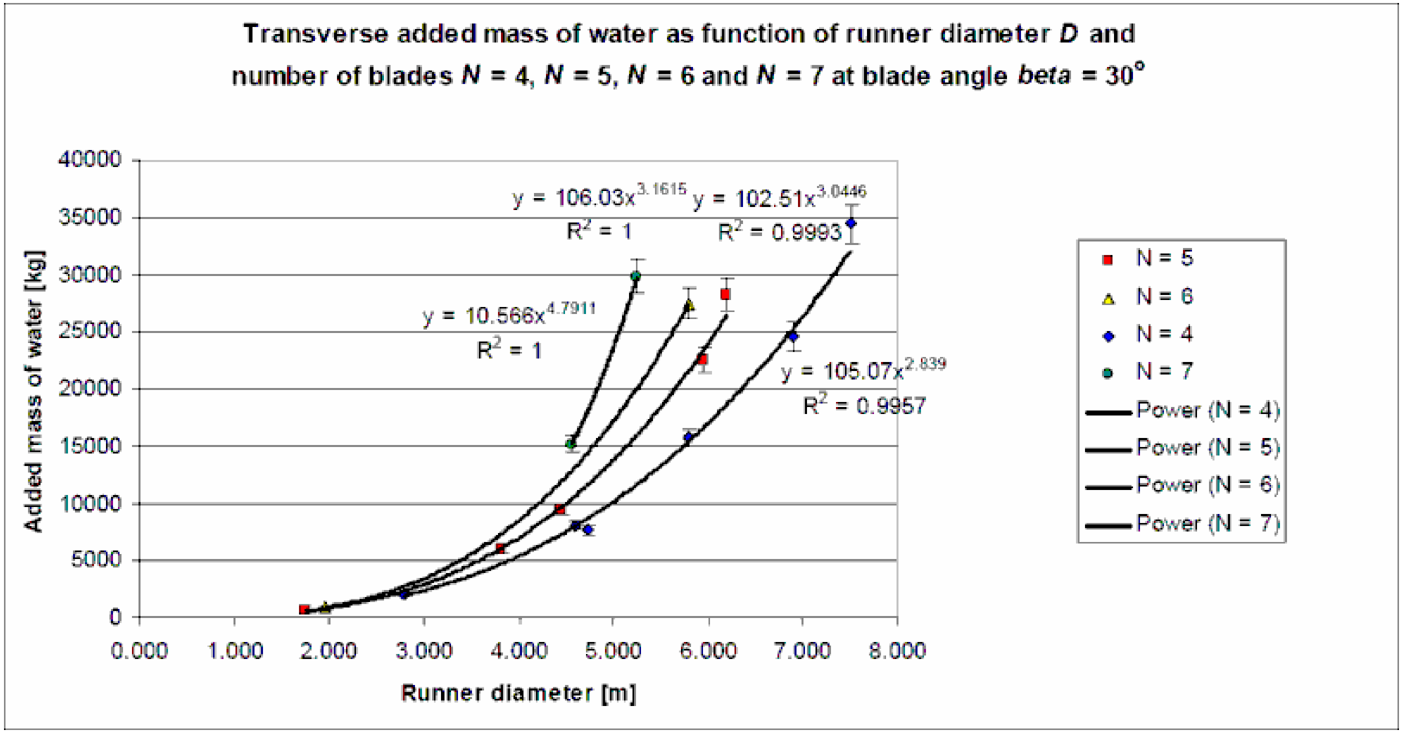



Rotor model in the Finite Element Method

The objective of calculations was to determine the entrained masses for different rotor sizes, different numbers of blades and different blade pitch settings. Calculations were performed for turbines having powers from 3 [MW] to 75 [MW] and rotor diameters from od 1.75 [m] to 7.5 [m].

The entrained mass of the rotor in transverse direction $M_{22}$ for the diameter 4.5 [m] at different blade pitch settings

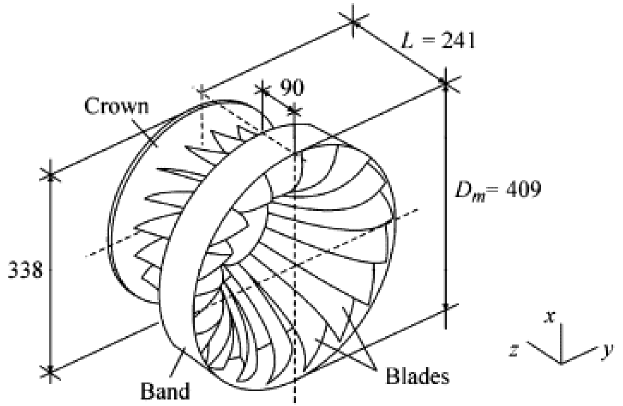


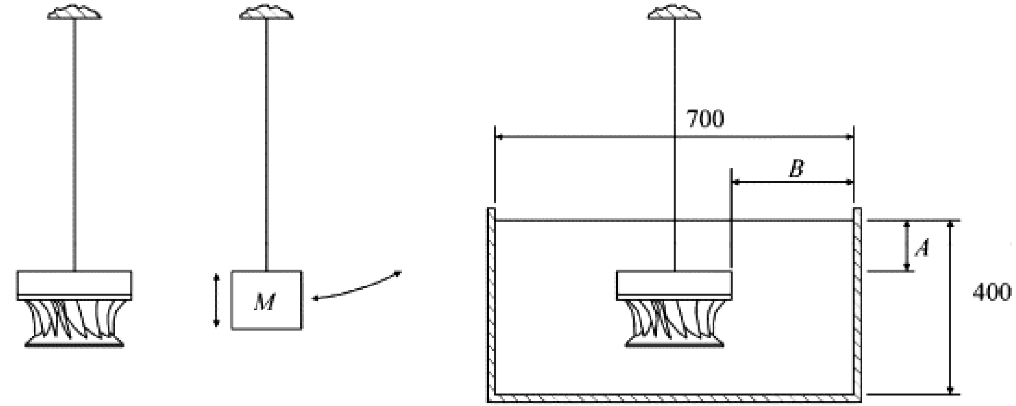

The entrained mass $M_{22}$ for rotors of different diameters and numbers of blades at the same pitch setting angle $\beta$

# Influence of the entrained mass on vibration of the Francis water turbine

## Model experiments




Model of the rotor




Test set-up for measurements in water and in air

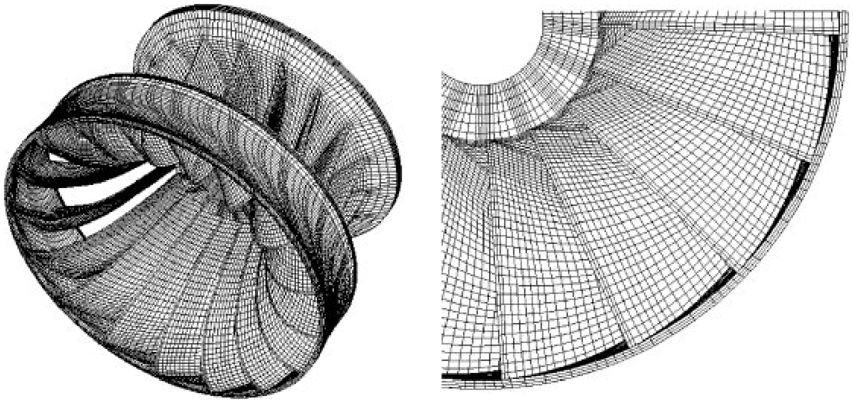
The rotor model was excited using a special inducer (hammer), in 118 selected points, exciting different modes of own vibration in air and in water. The vibrations were measured and registered using special sensoring system. One degree of freedom vibrations of the rotor may be described by the following equation:

$$\left(M_W + M_A\right)\ddot{X} + \left(C_W + C_A\right)\dot{X} + \left(K_W + K_A\right)X = F(t)$$

Index A denotes the effect of immersion of the rotor in water, concerning the mass M, the damping coefficient C and the stiffness coefficient K.

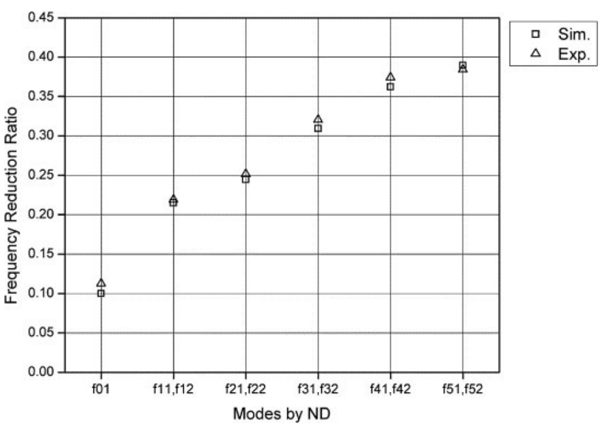
# Influence of the entrained mass on vibration of the Francis water turbine

## Numerical calculations

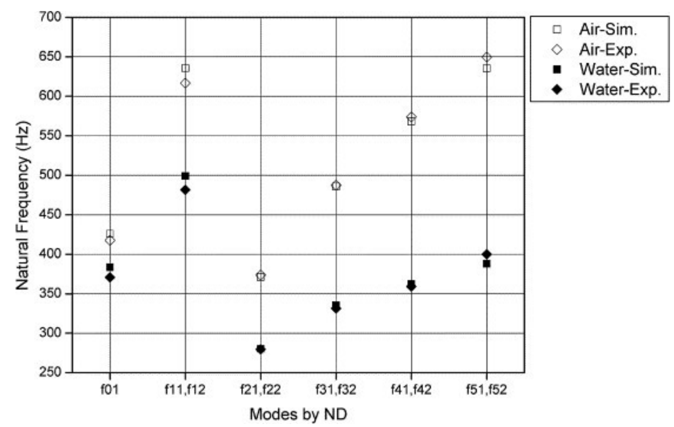


Model of the rotor for FEM calculation

Every geometrically repeatable sector of the rotor was modelled by 6133 hexahedral finite elements. The results of calculations were compared with experimental measurements discussed before. In all cases the immersion in water has reduced the own frequencies of vibration of the rotor.




Measured and calculated degree of reduction of frequency of the different vibration modes in water




Comparison of the calculated and measured frequencies of the different vibration modes in air and in water