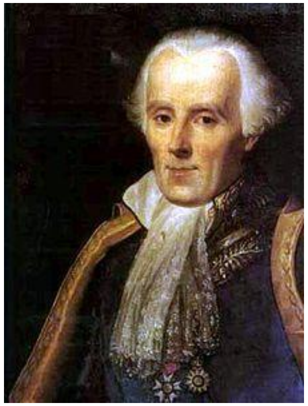
J. Szantyr – Lecture No. 9 – Computations of Potential Flows

If the fluid flow is irrotational, i.e. everywhere or almost everywhere there is $rot\bar{u} = 0$ then there exists a scalar function $\varphi(x, y, z, t)$ such that $\bar{u} = grad\varphi$. Such a flow is called a potential flow and the scalar function φ is called the velocity potential.

There is: $u_x = \frac{\partial \varphi}{\partial x} \quad u_y = \frac{\partial \varphi}{\partial y} \quad u_z = \frac{\partial \varphi}{\partial z}$



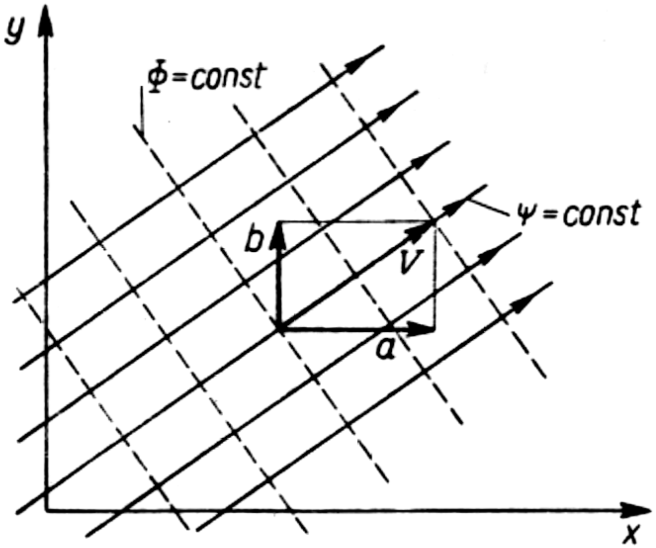
Pierre Laplace
1749 - 1827

In case of the potential flow of incompressible fluid the mass conservation equation transforms into the Laplace equation:

$$\frac{\partial \rho}{\partial t} + div(\rho\bar{u}) = 0 \rightarrow divgrad\varphi = \Delta\varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0$$

Laplace equation is linear, therefore a sum of its solutions is also a solution. Consequently, very complicated potential flow functions, describing complex flows, may be composed as the superposition of many simple functions, describing elementary flows.

Elementary flows – a uniform flow



$$u_x = a = \frac{\partial \varphi}{\partial x} \qquad u_y = b = \frac{\partial \varphi}{\partial y}$$

Velocity potential:

$$\varphi(x, y) = a \cdot x + b \cdot y = u_x \cdot x + u_y \cdot y$$

Equipotential lines:

$$y = -\frac{a}{b} x + C = -\frac{u_x}{u_y} x + C$$

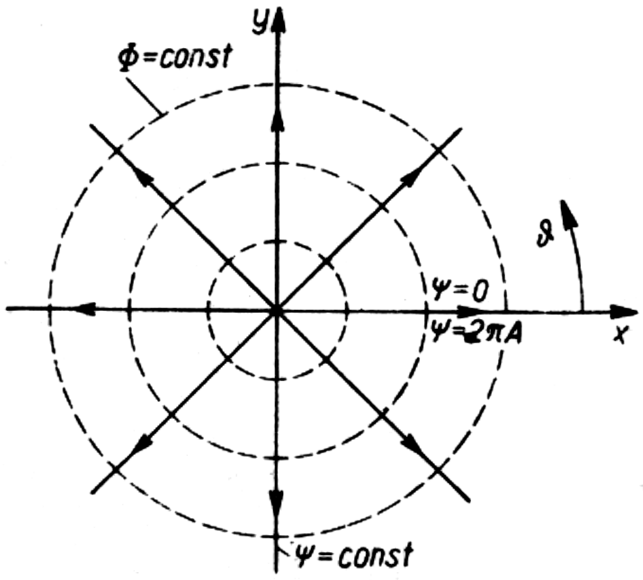
Stream function:

$$\psi(x, y) = a \cdot y - b \cdot x = u_x \cdot y - u_y \cdot x$$

Streamlines:

$$y = \frac{b}{a} x + C = \frac{u_y}{u_x} x + C$$

Elementary flows – a source (positive or negative)



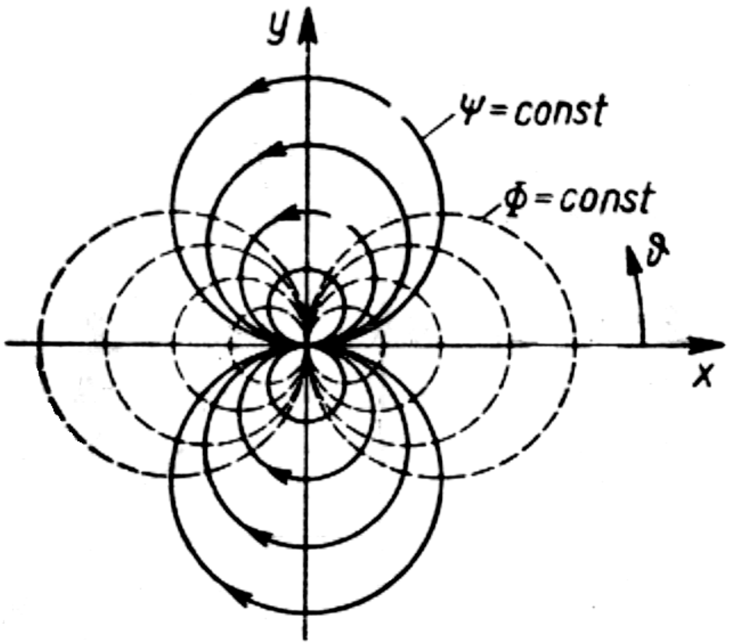
Source is a singular point in the field of flow, in which an outflow of the fluid with a defined volumetric intensity *Q* takes place. This outflow is uniform in all directions. In the case of a negative source (or a sink), the fluid flows towards the source and „disappears" in it. Hence we have:

$Q = \pm 2\pi r u_r$ or: $u_r = \pm \dfrac{Q}{2\pi r}$ where: u_r - radial velocity

$$u_r = \frac{\partial \varphi}{\partial r} = \pm \frac{Q}{2\pi r} \rightarrow \varphi = \pm \frac{Q}{2\pi} \ln r$$

Constant values of the potential φ appear for constant values of the radius *r*, hence the equipotential lines are concentric circles.

Elementary flows – a double source (a dipole)



Dipole is the effect of superposition of positive and negative sources of the same module of flow intensity. Intensity of a dipole is measured by the so called moment of the dipole $M=2aQ$. <u>As opposed to the source, the dipole has directional characteristics, because it ejects fluid in a definite direction. Therefore the orientation of the dipole in space is important.</u>

For a dipole at $x=0$, $y=0$ directed along positive x axis we have:

Dipole potential:

$$\varphi = -\frac{M}{2\pi}\frac{x}{x^2+y^2}$$

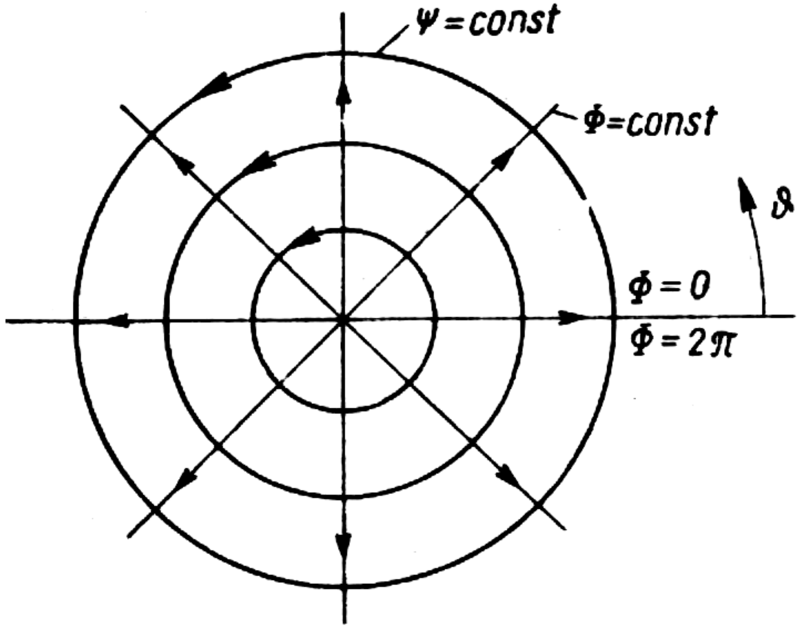
Dipole stream function:

$$\psi = \frac{M}{2\pi}\frac{y}{x^2+y^2}$$

In order to generate hydrodynamic forces on objects in potential flow an asymmetrical flow must be generated on these objects. This is possible by using another elementary flow called a vortex.

Elementary flows – a vortex

A vortex is a singular point generating in its vicinity a flow with circular trajectories.



Vortex is called a transformed source, because the streamlines of the vortex coincide with equipotential lines of the source and vice versa.

Vortex potential: $\varphi = A \cdot \theta$ Vortex stream function: $\psi = A \cdot \ln r$

$$u_r = \frac{\partial \varphi}{\partial r} = 0 \qquad u_\theta = \frac{1}{r}\frac{\partial \varphi}{\partial \theta} = \frac{A}{r}$$

The constant A is connected with the velocity circulation along a closed contour encompassing the vortex:

$$\Gamma_C = \oint_C \bar{u} \bullet d\bar{s} = \int_0^{2\pi} \bar{u} \bullet d\bar{s} = \int_0^{2\pi} \frac{A}{r} \bullet r d\theta = \int_0^{2\pi} A d\theta = 2\pi A \rightarrow A = \frac{\Gamma_C}{2\pi}$$

Hence we obtain: $$u_\theta = \frac{\Gamma}{2\pi r}$$

It should be noticed that the fluid motion generated by the vortex is irrotational in the entire space outside the vortex itself. Calculation of circulation along a contour not encompassing the vortex produces zero. Consequently, we have an isolated vortex at *x=0, y=0* and irrotational flow around it. This gives us the possibility to treat the entire flow domain as potential flow.

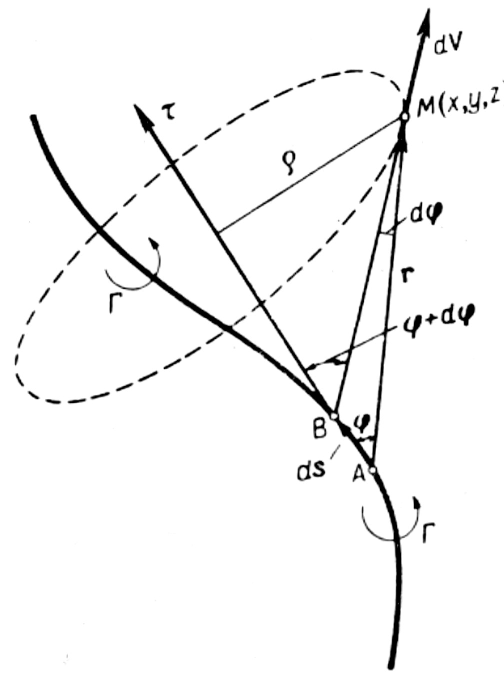
In practical modelling the flow domain may be divided into irrotational and rotational regions. Both these regions are interdependent. The rotational flow region may be modelled by vortex filaments. Then it becomes important to calculate the velocity field generated by the vorticity field, i.e. inverse operation to calculation of rotation of the velocity field.

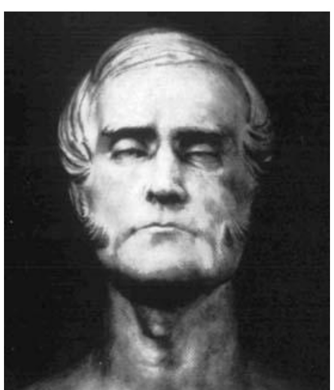
Biot-Savart formula

Jean Baptiste Biot
1774 - 1862

$$d\overline{V} = \frac{\Gamma}{4\pi} \frac{d\overline{s} \times \overline{r}}{r^3}$$

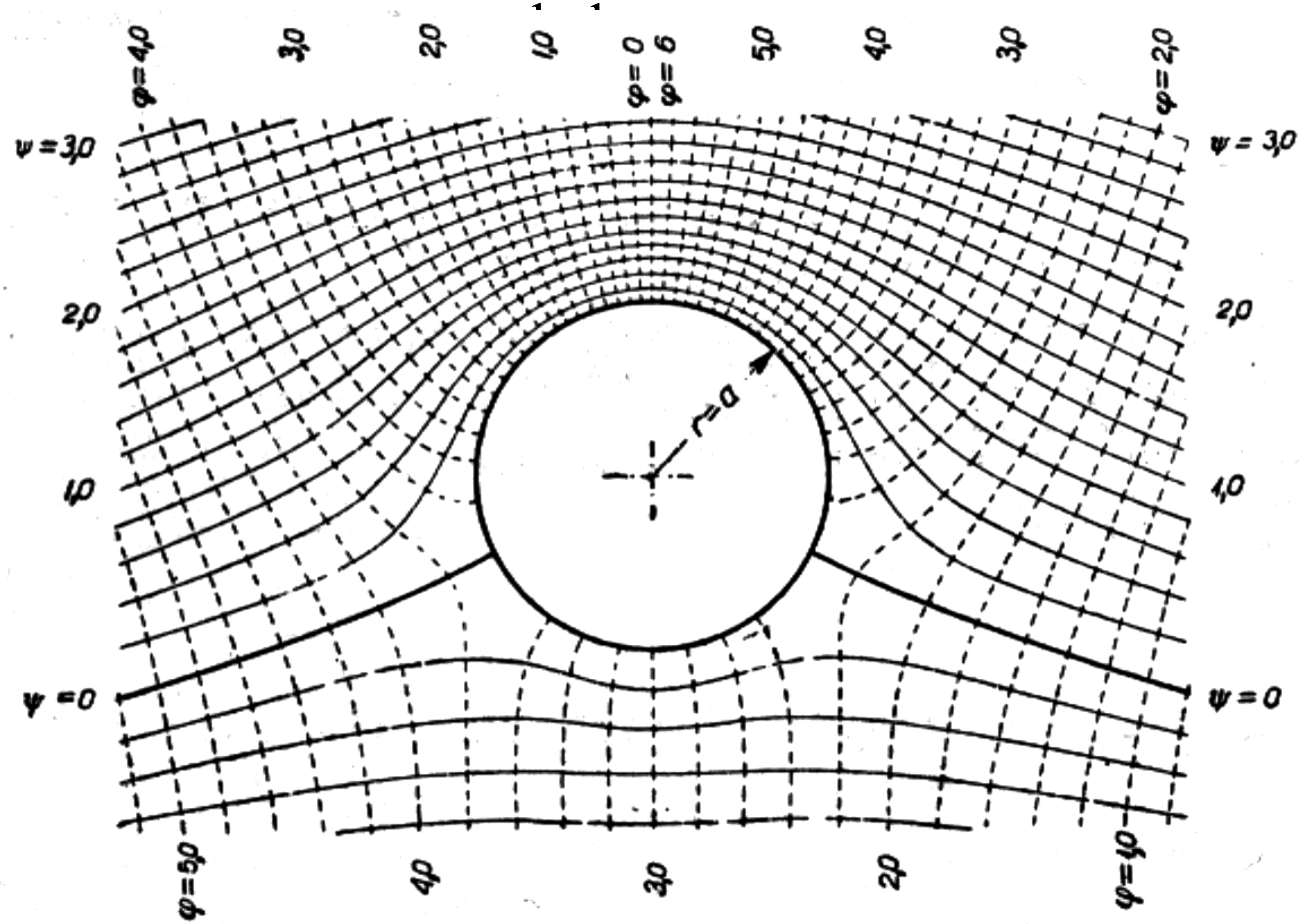
$$\overline{V} = \frac{\Gamma}{4\pi} \int_L \frac{d\overline{s} \times \overline{r}}{r^3}$$





Felix Savart
1791 - 1841

Example: vortex (rotational) flow around a cylinder



Superposition of the uniform flow with dipole and vortex located at the origin of the system of co-ordinates.

Potential: $$\varphi = u_\infty\left(r+\frac{a^2}{r}\right)\cos\theta - \frac{\Gamma}{2\pi}\theta$$ where: $$\theta = arctg\,\frac{y}{x}$$

Stream function: $$\psi = u_\infty\left(r-\frac{a^2}{r}\right)\sin\theta + \frac{\Gamma}{2\pi}\ln\frac{r}{a}$$

Velocity components: $$u_r = u_\infty\left(1-\frac{a^2}{r^2}\right)\cos\theta$$

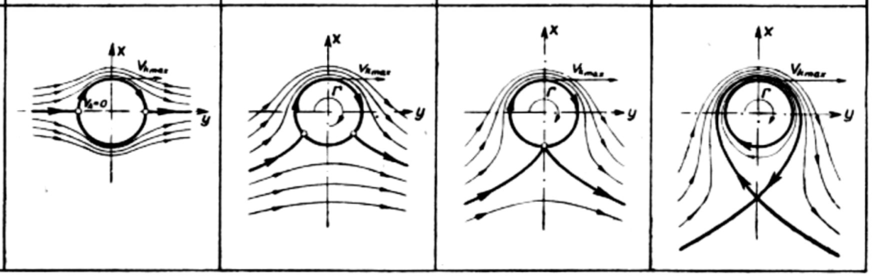
$$u_\theta = -u_\infty\left(1+\frac{a^2}{r^2}\right)\sin\theta - \frac{\Gamma}{2\pi r}$$

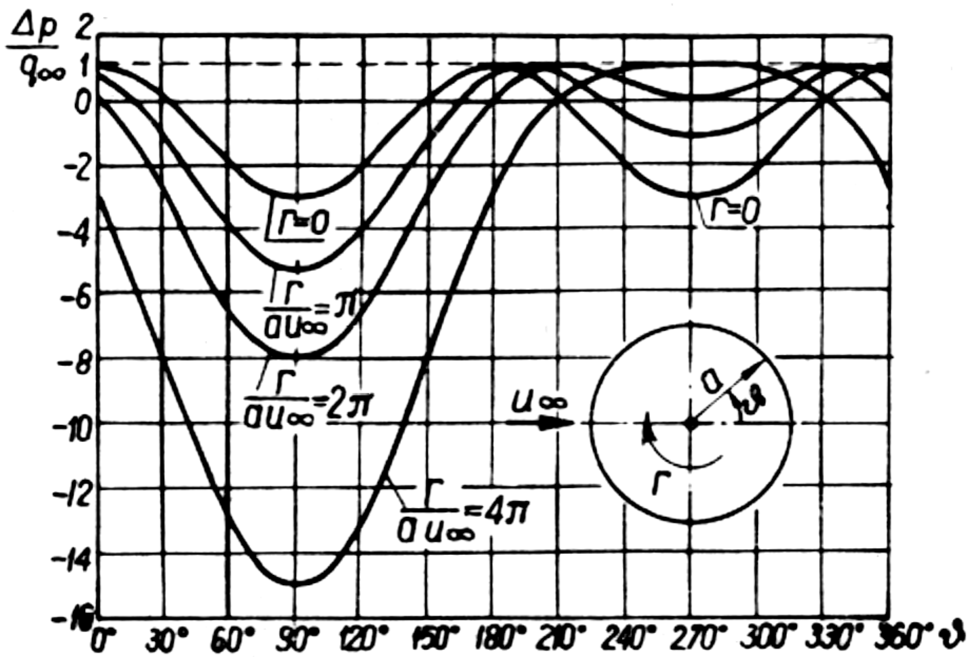
On the cylinder surface there is: $$u_r = 0 \qquad u_\theta = -u_\infty\left(2\sin\theta + \frac{\Gamma}{2\pi a u_\infty}\right)$$

Pressure distribution on the cylinder from Bernoulli:

$$p_\theta = p_\infty + \frac{\rho u_\infty^2}{2}\left(1-\frac{u_\theta^2}{u_\infty^2}\right)$$

Γ	0	$2\pi a u_\infty$	$4\pi a u_\infty$	$> 4\pi a u_\infty$
$V_{k_{max}}$	$2u_\infty$	$3u_\infty$	$4u_\infty$	
$\sin\vartheta = -\frac{\Gamma}{4\pi a u_\infty}$	0	$-\frac{1}{2}$	-1	
ϑ dla $V_k = 0$	0 π	$\frac{7}{6}\pi$ $\frac{4}{6}\pi$	$\frac{3}{2}\pi$	





Asymmetrical pressure distribution round the cylinder depends on the value of circulation. For determination of circulation an additional condition is necessary, specifying the location of the stagnation point on the cylinder.

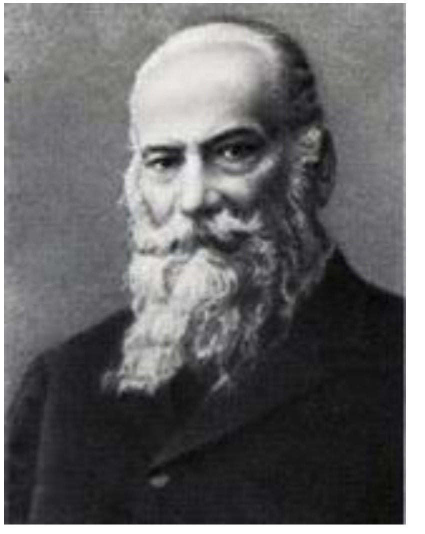
The pressure distribution on the cylinder surface may be determined in the form of a non-dimensional coefficient:

$$C_p = \frac{p_\theta - p_\infty}{\frac{1}{2}\rho u_\infty^2} = 1 - \left(2\sin\theta + \frac{\Gamma}{2\pi a u_\infty} \right)^2$$

On the basis of the pressure distribution the components of the resultant hydrodynamic force may be determined:

$$P_x = -a\int_0^{2\pi} p_\theta \cos\theta d\theta = 0$$ - drag force

$$P_y = -a\int_0^{2\pi} p_\theta \sin\theta d\theta = \rho u_\infty \Gamma$$ - lift force

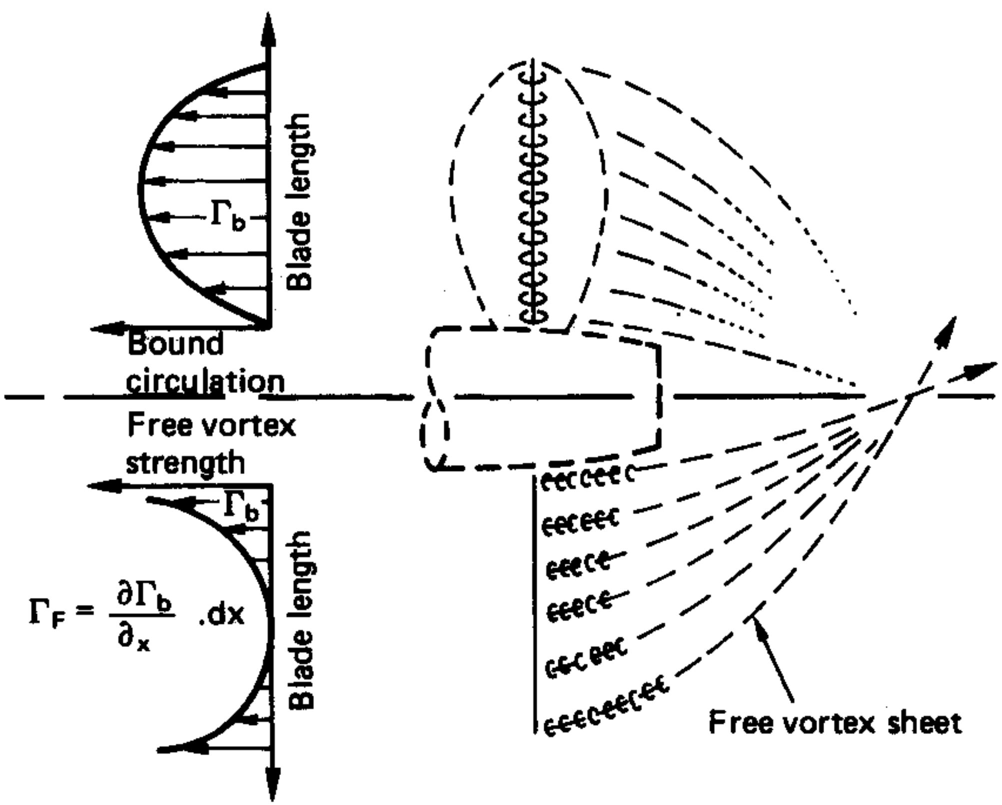


Nicolai Joukovsky
1847 - 1921

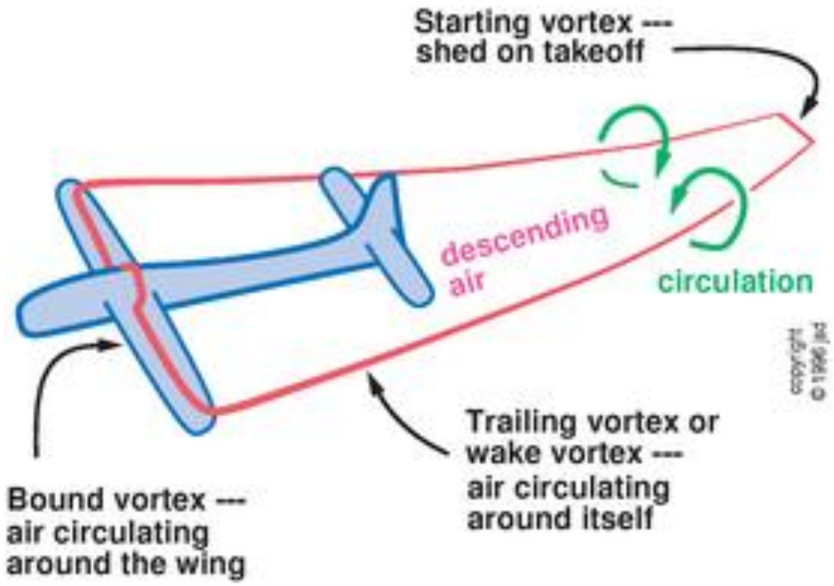
Joukovsky theorem: lift force acting on the unit length of the cylinder is equal to the multiple of fluid density, undisturbed flow velocity and velocity circulation around the cylinder.

Contemporary methods for determination of potential flows

- lifting line method
- lifting surface method
- boundary elements method



Lifting line method is based on modelling the lifting foil with a single vortex line, called the bound vortex, which generates lift according to Joukovsky theorem. This vortex must be substituted with the system of free vortices. Lifting line method is suitable for modelling foils with high aspect ratio, i.e. airplane wings and air screw blades.

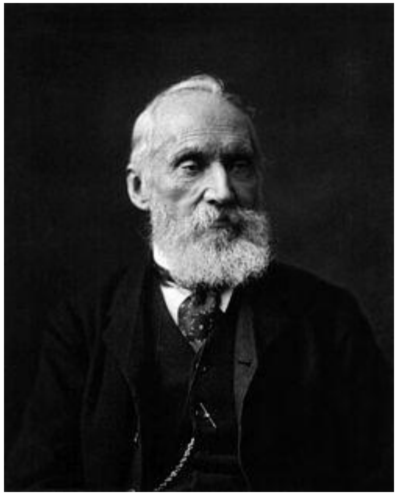


Joukovsky theorem may be employed e.g. for determination of the lift force on the aircraft wing, according to the relation:

$$\overline{L} = \rho \overline{U}_{\infty} \times \overline{\Gamma}$$

The above relation determines not only the magnitude, but also the direction of the lift force. In the potential flow the vortex lines cannot terminate inside the fluid (cf. theorems on the following slide). Hence it is necessary to supplement the bound vortex with the system of free vortices, which contribute to the velocity field calculation according to the Biot-Savart formula.

Thomson theorem: in the flow of an ideal barotropic fluid taking place in the field of potential mass forces the circulation along an arbitrary closed fluid line does not change with time.

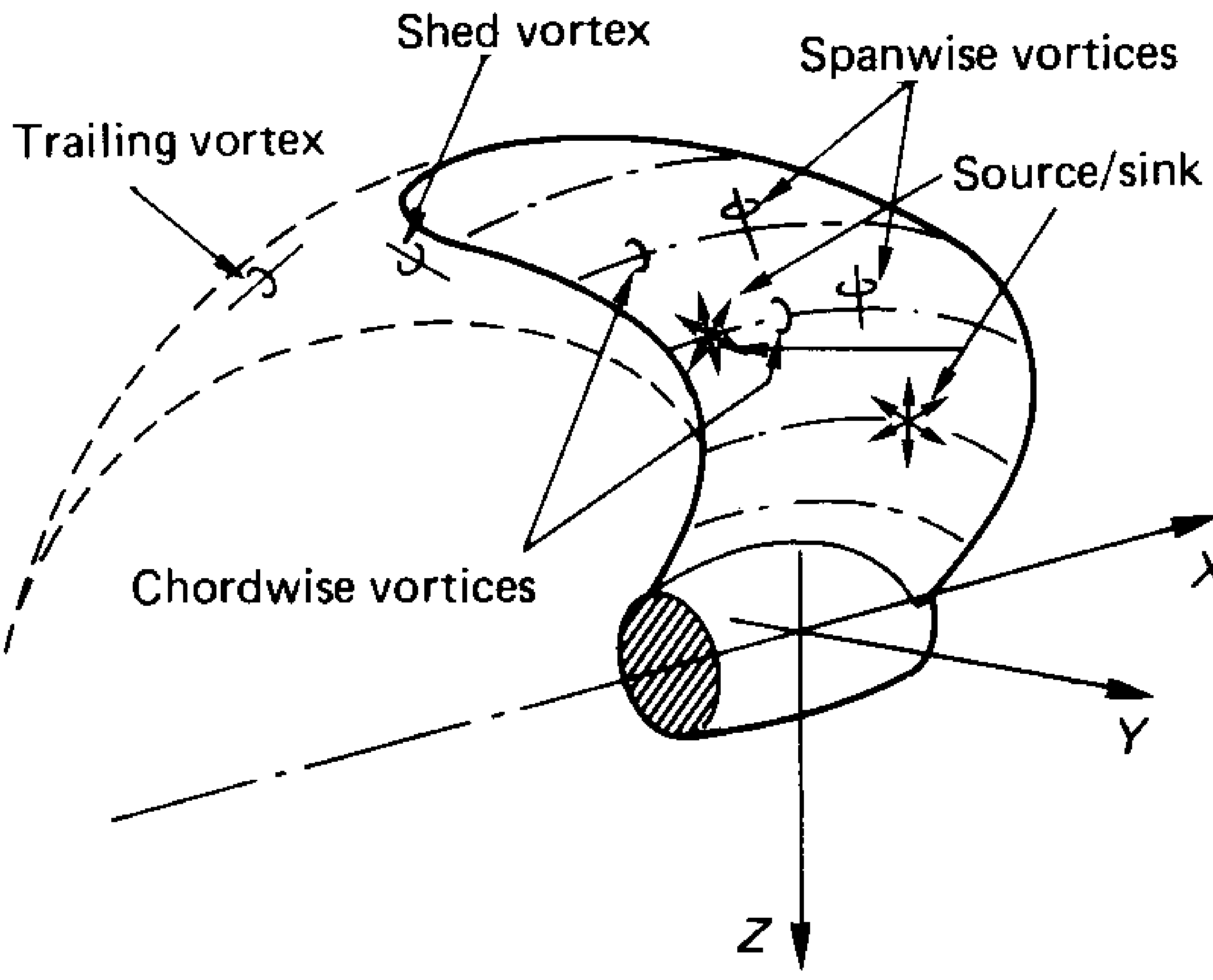


Wiliam Thomson lord Kelvin
1824 - 1907

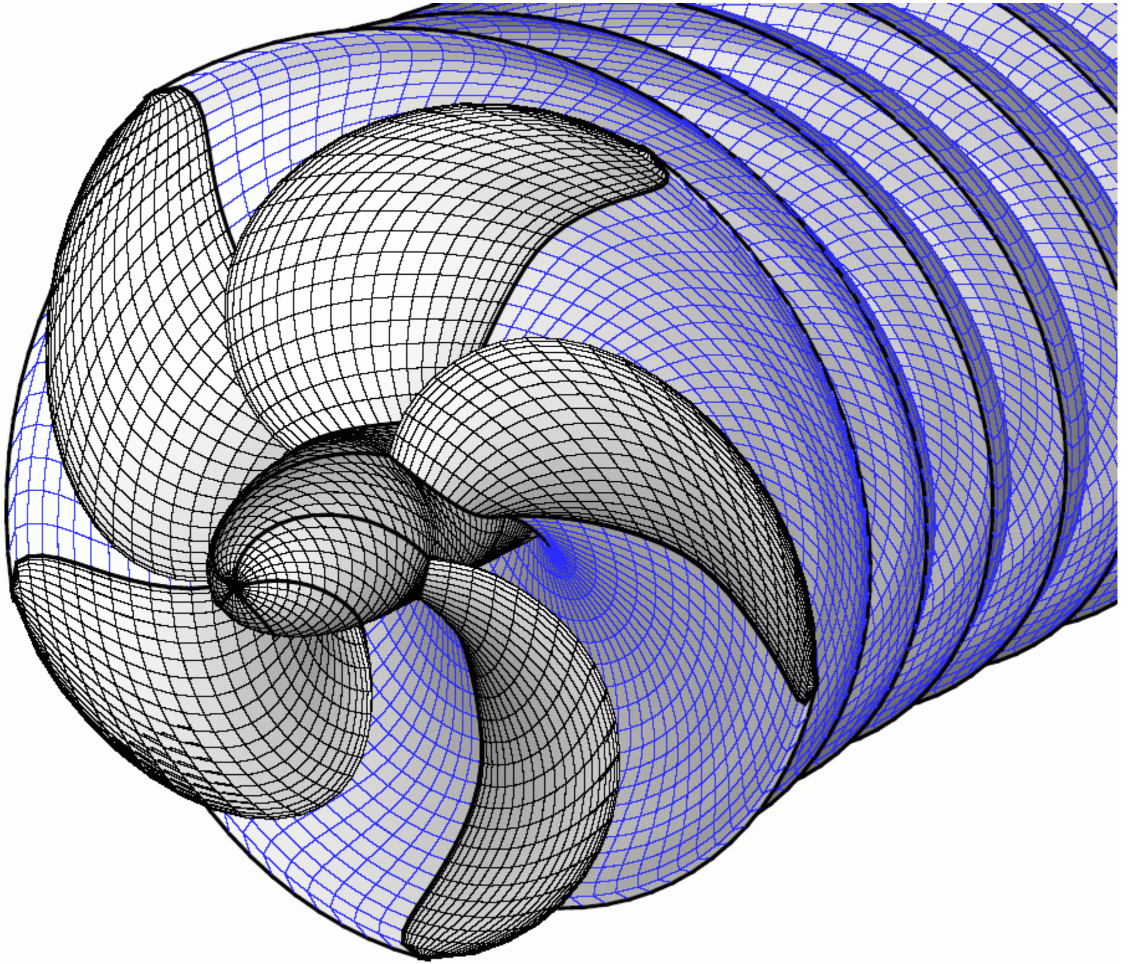
Second Helmholtz theorem: In the flow of an ideal barotropic fluid taking place in the field of potential mass forces the intensity of a vortex filament does not change along its length and remains constant in time.

Hermann von Helmholtz
1821 - 1894

Lifting surface method is based on distribution of the vortices, dipoles and sources on an infinitely thin surface bound by the true foil outline. This method is suitable for modelling of foils with low aspect ratio, e.g. marine propeller blades, turbine blades etc.



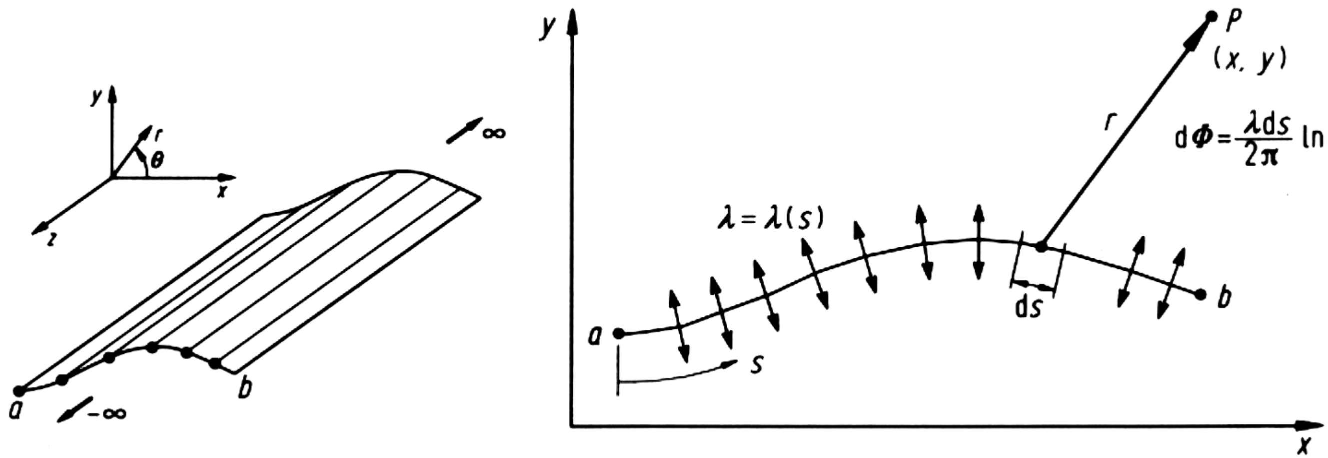
Boundary elements method is based on distributing the vortices, dipoles and sources on the true surface of the modelled object, i.e. on both sides of an aircraft wing or turbine blade. This method is suitable for determination of flow around complicated objects e.g. whole aircraft, ships or vehicles. Many thousands of elements may be used in such modelling.



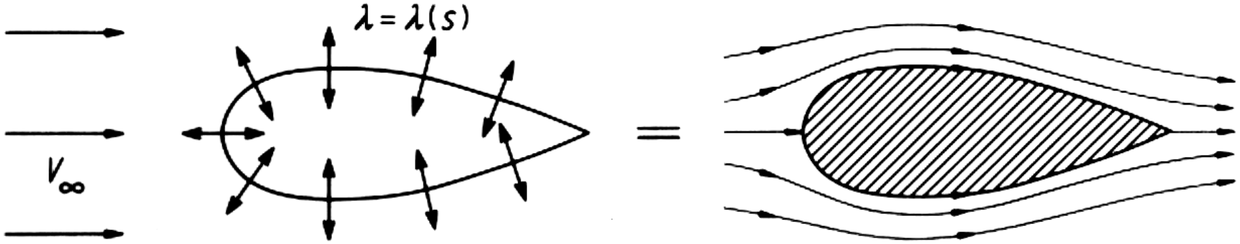
Pressure distribution on a marine propeller, computed using the boundary elements method

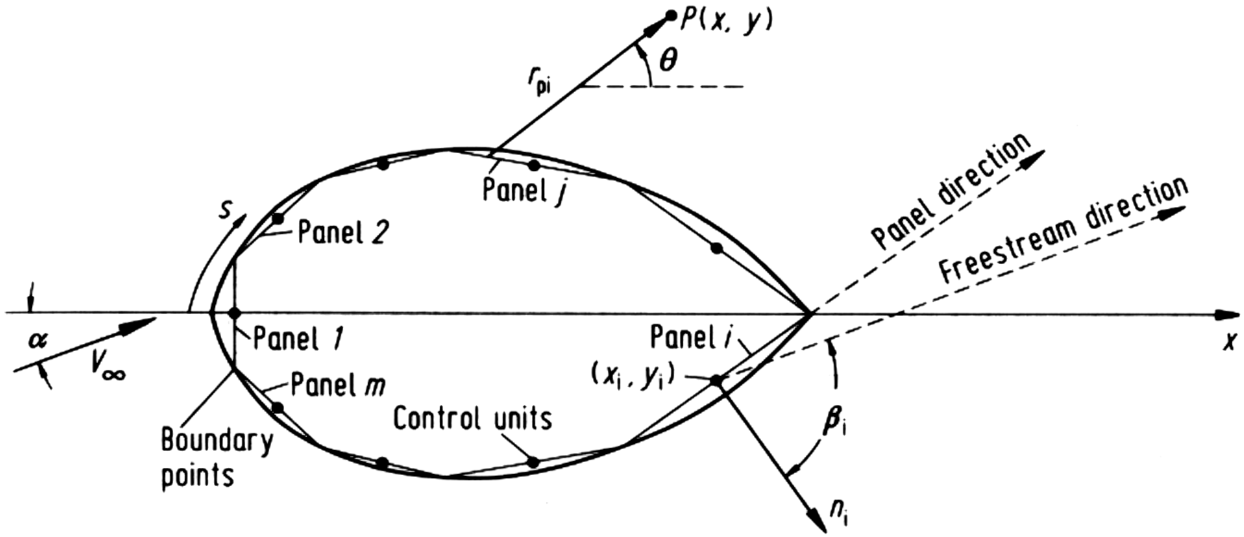
Geometrically complicated objects may be modelled using continuous distributions of sources, vortices and dipoles. E.g. a continuous distribution of sources along the curve a-b may be described by the potential:

$$\varphi(x,y)=\int_a^b \frac{\lambda ds}{2\pi}\ln r$$ where: $\lambda[m/s]$ - continuous distribution of sources



In practice the geometrically complicated surface is divided into a number of elements called panels. In two-dimensional flows panels are usually sections of a straight line, in three-dimensional flows they are sections of a flat surface.



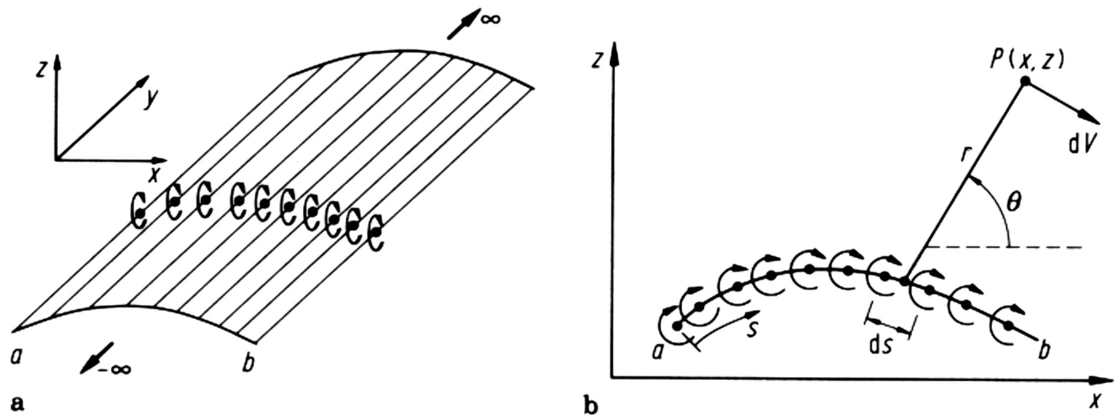


Solution of such a flow is based on the boundary condition, which postulates no flow through the surface of the object, i.e. zero normal velocity on the surface. This leads to the following equation:

$$\frac{\lambda_i}{2} + \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{\lambda_j}{2\pi} \int_j \frac{\partial}{\partial n_j} \left(\ln r_{ij} \right) ds_j + V_\infty \cos \beta_i = 0$$

Formulation of such an equation for every panel leads to the system of linear equations for the unknown intensities of sources λ.

Modelling of flows with lift forces requires distributions of vortices and dipoles. In case of the continuous distribution of vortices with intensity γ we have:



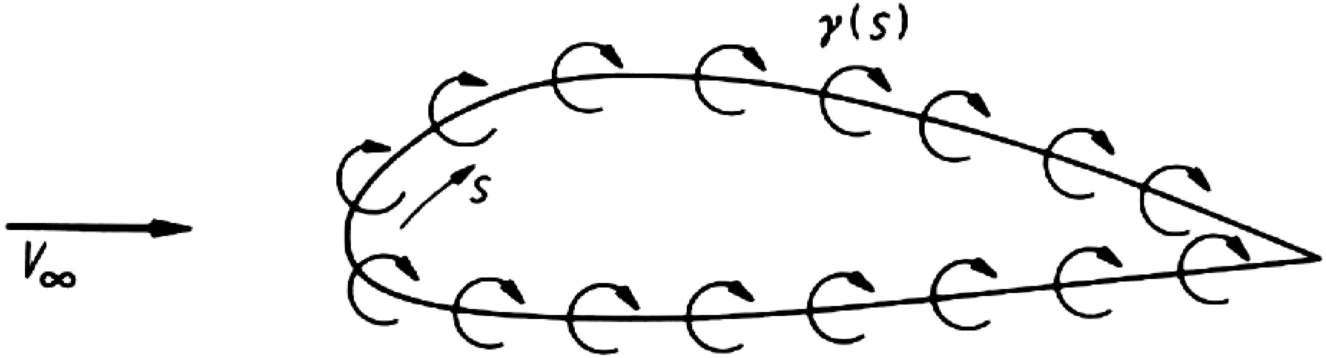
In this case the flow is described by the potential:

$$\varphi(x, y) = -\frac{1}{2\pi}\int_a^b \theta\gamma ds$$

where: $\gamma[m/s]$ - continuous distribution of vortices

After the object is divided into panels we obtain the following equation:

$$V_\infty \cos\beta_i - \sum_{j=1}^{n}\frac{\gamma_j}{2\pi}\int_j \frac{\partial\theta_{ij}}{\partial n_i} ds_j = 0$$



In case of modelling the object which generates lift force by means of the vortex distribution, an additional condition is necessary for univocal determination of the vortices intensity. For a profile the so called Kutta condition is most frequently used. This condition postulates the flow to leave the profile tangentially, precisely at the trailing edge. After the system of equations is solved and the intensity of all vortices is determined, the lift force may be calculated from the Joukovsky theorem:

$$L = \rho V_\infty \sum_{j=1}^{n} \gamma_j s_j$$

Martin Kutta
1867 - 1944