J. Szantyr – Lecture No. 11 – Navier-Stokes equation

Substitution of the relations resulting from the Newtonian fluid model into the equation of conservation of the fluid momentum leads to the equation known as the **Navier-Stokes equation.**

This equation may be written in the form of three scalar equations:

$$\rho \frac{Du}{Dt} = \rho f_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[2\mu \frac{\partial u}{\partial x} + \lambda div\bar{u}\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right]$$

$$\rho \frac{Dv}{Dt} = \rho f_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[2\mu \frac{\partial v}{\partial y} + \lambda div\bar{u}\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right]$$

$$\rho \frac{Dw}{Dt} = \rho f_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[2\mu \frac{\partial w}{\partial z} + \lambda div\bar{u}\right]$$

In the vector form the Navier-Stokes equation reads:

$$\rho \frac{D\bar{u}}{Dt} = \rho\bar{f} - grad p + grad\left(\lambda div \bar{u}\right) + div\left(2\mu\left[D\right]\right)$$

$$A = B + C + D + E$$

A – rate of change of the fluid element momentum

B- mass force

C- surface pressure force

D – surface force connected with fluid viscosity and resulting from the change of volume of the <u>compressible</u> fluid element (compression or expansion)

E- surface force connected with fluid viscosity and resulting from the linear and shearing deformation of the fluid element

In an incompressible fluid the Navier-Stokes equation simplifies to the form:

$$\rho \frac{D\bar{u}}{Dt} = \rho\bar{f} - grad p + div(2\mu[D])$$

If additionally a constant fluid viscosity is assumed, we obtain:

$$\rho \frac{D\bar{u}}{Dt} = \rho\bar{f} - grad p + \mu\Delta\bar{u}$$

Further possible simplification is the assumption of zero viscosity of the fluid, which leads to the **Euler equation**, describing the motion of an incompressible and inviscid fluid:

$$\rho \frac{D\bar{u}}{Dt} = \rho\bar{f} - grad p$$

The Navier-Stokes equation may be solved analytically only for a few simplified cases. Selected examples are described below.

Examples of the analytical solutions of the Navier-Stokes for simple flows

Assumption: we consider an unidirectional flow, i.e. the flow in which *v=w=0*, so the velocity vectors are parallel to each other in every point of the field.

If the fluid is incompressible, we get from the mass conservation equation:

$$\frac{\partial u}{\partial x} = 0 \qquad \text{or:} \qquad u = u(y, z, t)$$

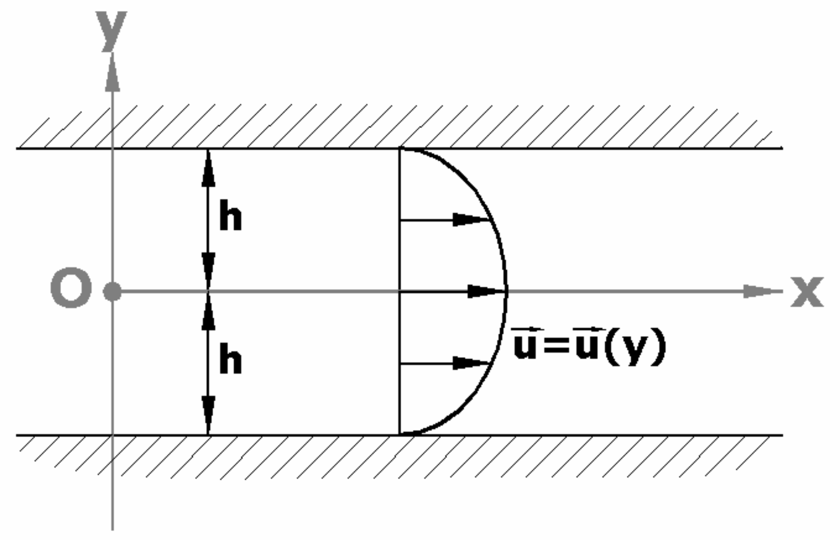
Then the Navier-Stokes equation simplifies to the form:

$$\rho \frac{\partial u}{\partial t} - \mu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = -\frac{\partial p}{\partial x}$$

As the left hand side does not depend on x, and: $\frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0$

then: $p = p(x, t)$ $\quad \frac{dp}{dx} = f(t)$ and finally: $\frac{dp}{dx} = \frac{\Delta p}{\Delta x}$

Example No. 1: Steady laminar flow between two infinite parallel plates (Poiseuille flow)



Given: $\frac{\Delta p}{\Delta x} = const$

Boundary conditions:

u=0 for *y=h*

u=0 for *y=-h*

Solution

The Navier-Stokes equation takes the form: $$\frac{d^2 u}{dt^2} = \frac{1}{\mu}\frac{\Delta p}{\Delta x}$$

After double integration we obtain:

$$u(y) = \frac{1}{\mu}\frac{\Delta p}{\Delta x}\frac{y^2}{2} + C_1 y + C_2$$

The integration constants may be determined from the boundary conditions, what leads to the solution:

$$u(y) = -\frac{1}{2\mu}\frac{\Delta p}{\Delta x}\left(h^2 - y^2\right)$$

The volumetric flow intensity may be determined as follows:

$$Q = \int_S u dS = \int_{-h}^{+h} u(y) dy = -\frac{2}{3\mu}\frac{\Delta p}{\Delta x} h^3$$

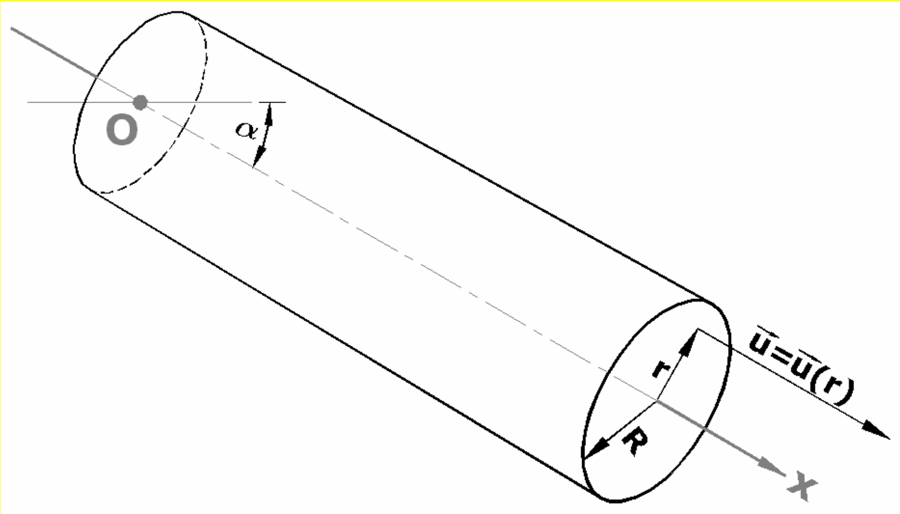
Comments:

-the velocity profile is parabolic with maximum at y=0,

-for increasing pressure gradient the maximum velocity and the volumetric flow intensity also increase,

-for increasing fluid viscosity the maximum velocity and the volumetric intensity of flow decrease.

Example No. 2: Steady laminar flow through the horizontal pipe of constant circular cross-section (case of α=0)



Given: $\frac{\Delta p}{\Delta x} = const$

Boundary conditions:

$$u(R) = 0 \qquad u(0) < \infty$$

Solution

We employ the cylindrical system of co-ordinates, in which the Laplace operator applied to the velocity field has the form:

$$\nabla^2 u = \frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)$$

The Navier-Stokes equation is: $$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)=\frac{1}{\mu}\frac{\Delta p}{\Delta x}$$

Double integration with respect to r gives:

$$r\frac{du}{dr}=\frac{1}{\mu}\frac{\Delta p}{\Delta x}\frac{r^2}{2}+C_1 \qquad u=\frac{1}{\mu}\frac{\Delta p}{\Delta x}\frac{r^2}{4}+C_1\ln r+C_2$$

The integration constants may be determined from the boundary conditions:

$$C_1=0 \qquad C_2=-\frac{1}{\mu}\frac{\Delta p}{\Delta x}\frac{R^2}{4}$$

Ultimately this leads to:

$$u(r)=-\frac{1}{4\mu}\frac{\Delta p}{\Delta x}\left(R^2-r^2\right)$$

$$Q=\int_0^R u(r)\cdot 2\pi r dr=-2\pi\cdot\frac{1}{4\mu}\frac{\Delta p}{\Delta x}\int_0^R\left(R^2-r^2\right)dr=-\frac{\pi}{8\mu}\frac{\Delta p}{\Delta x}R^4$$

Comment: comparison with the Example No. 1 assuming h=R leads to the conclusion that with the same pressure gradient the maximum flow velocity and the volumetric flow intensity in the pipe are smaller. This is due to the more intensive retardation of the pipe flow by the viscous stresses.

<u>The case of pipe inclined at an angle α to the horizon</u>

In this case the Navier-Stokes equation must include the component of the mass force acting in the direction of flow:

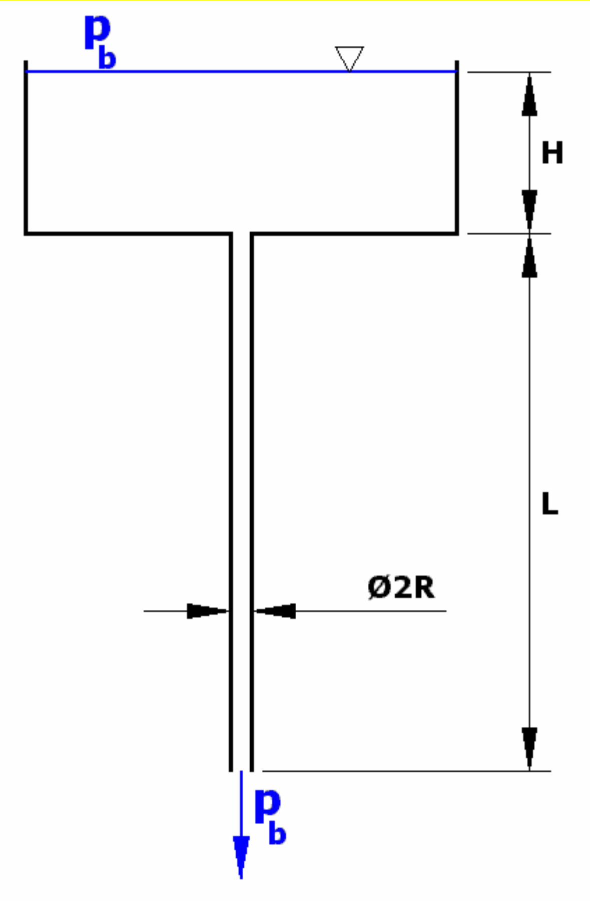
$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)=\frac{1}{\mu}\frac{\Delta p}{\Delta x}-\frac{g}{\nu}\sin\alpha$$

As the pressure gradient and the mass flow component act on the flow in a similar way we may introduce the hydraulic head J:

$$J=\sin\alpha-\frac{1}{\rho g}\frac{\Delta p}{\Delta x}$$

Then the solution has the form:

$$u(r) = \frac{gJ}{4\nu}\left(R^2 - r^2\right) \qquad Q = \frac{\pi g J}{8\nu} R^4$$



<u>A particular case: the vertically oriented pipe</u>

In this case we have: $J = 1 - \frac{\Delta p}{\rho g L}$

Pressure at inlet: $p_1 = p_b + \rho g H$

Pressure at outlet: $p_2 = p_b$

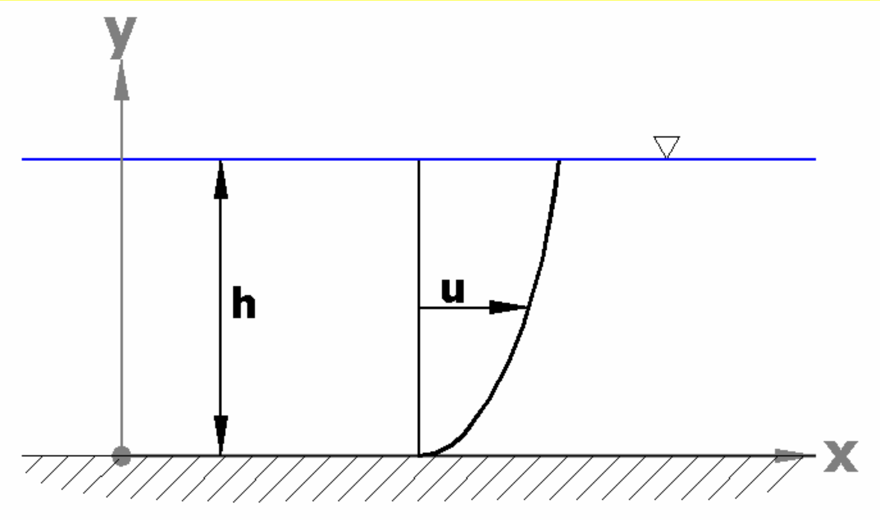
Then: $\Delta p = p_2 - p_1 = -\rho g H$

After substitution we get:

$$Q = \frac{\pi g R^4}{8\nu}\left(1 + \frac{H}{L}\right)$$

Due to the easy measurement of Q this formula may be employed for experimental determination of the fluid viscosity coefficient ν

Example No. 3: Flow in an open channel



Boundary conditions:

$$\frac{du}{dy} = 0 \qquad \text{for: } y=h \qquad (1)$$

$$u=0 \qquad \text{for } y=0 \qquad (2)$$

Solution

$$\frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{\Delta p}{\Delta x} \quad \rightarrow \quad \frac{du}{dy} = \frac{1}{\mu}\frac{\Delta p}{\Delta x} y + C_1$$

From condition (1): $C_1 = -\frac{1}{\mu}\frac{\Delta p}{\Delta x} h$ what leads to:

$$\frac{du}{dy} = \frac{1}{\mu}\frac{\Delta p}{\Delta x} y - \frac{1}{\mu}\frac{\Delta p}{\Delta x} h$$

After second integration we get: $u = \frac{1}{2\mu}\frac{\Delta p}{\Delta x}y^2 - \frac{1}{2\mu}\frac{\Delta p}{\Delta x}hy + C_2$

$C_2 = 0$ from condtion (2), what ultimately leads to:

$$u = \frac{1}{2\mu}\frac{\Delta p}{\Delta x}y\left(y - 2h\right)$$

And the volumetric intensity of flow is:

$$Q = \int_0^h \frac{1}{2\mu}\frac{\Delta p}{\Delta x}\left(y^2 - 2hy\right)dy = -\frac{2}{3}\frac{\Delta p}{\Delta x}h^3$$

NB!: when comparing the above results with those for the Poiseuille flow the differences in the adopted systems of co-ordinates should be taken into account.