J. Szantyr – Lecture No. 12 – Energy conservation equation

Kinetic energy of the fluid may be treated as the sum of the macroscopic motion energy and molecular motion energy or internal energy: $\int_V \left(\frac{u^2}{2} + e \right) dV$ $\qquad \bar{u}\left(u_x, u_y, u_z\right)$ $\qquad u = \left|\bar{u}\right|$

Rate of change (i.e. material derivative) of the total kinetic energy of the fluid volume *V* surrounded by the fluid surface *S* is equal to the sum of the power of mass forces, the power of surface forces and the stream of energy (heat) supplied to the element.

$$\frac{D}{Dt}\int_V \rho \left(\frac{u^2}{2} + e \right) dV = \int_V \rho \bar{f} \bullet \bar{u}\, dV + \int_{S(V)} \bar{\tau} \bullet \bar{u}\, dS - \int_{S(V)} \bar{j} \bullet \bar{n}\, dS$$

where:

	$\bar{f}$	unit mass force	$\bar{f}\left(f_x, f_y, f_z\right)$
	$\bar{\tau}$	unit surface force	
	$\bar{j}$	stream of supplied energy (heat)	$\bar{j}\left(j_x, j_y, j_z\right)$
	$\bar{n}$	External unit normal vector	

The surface integrals in this equation may be converted into the volume integrals. If simultaneously we express the unit surface forces by the stress tensor [**P**] , the above equation may be converted into the following form:

$$\int_V \left[\rho \frac{D}{Dt}\left(\frac{u^2}{2}+e\right) - \rho\bar{f} \bullet \bar{u} - div\left([P]\bar{u}\right) + div\bar{j} \right] dV = 0$$

As the volume of integration is arbitrarily selected, the function under the integral must be also equal to zero, what leads to the integral form of the energy conservation equation:

$$\rho \frac{D}{Dt}\left(\frac{u^2}{2}+e\right) = \rho\bar{f} \bullet \bar{u} + div\left([P]\bar{u}\right) - div\bar{j}$$

Full development of the operators in the equation leads to the following form:

$$\rho \frac{D}{Dt}\left(\frac{u^2}{2}+e\right)=\rho\left(f_x u_x + f_y u_y + f_z u_z\right)+\frac{\partial}{\partial x}\left[\left(-p+\tau_{xx}\right)u_x+\tau_{yx}u_y+\tau_{zx}u_z\right]+$$

$$+\frac{\partial}{\partial y}\left[\tau_{xy}u_x+\left(-p+\tau_{yy}\right)u_y+\tau_{zy}u_z\right]+\frac{\partial}{\partial z}\left[\tau_{xz}u_x+\tau_{yz}u_y+\left(-p+\tau_{zz}\right)u_z\right]+$$

$$-\frac{\partial j_x}{\partial x}-\frac{\partial j_y}{\partial y}-\frac{\partial j_z}{\partial z}$$

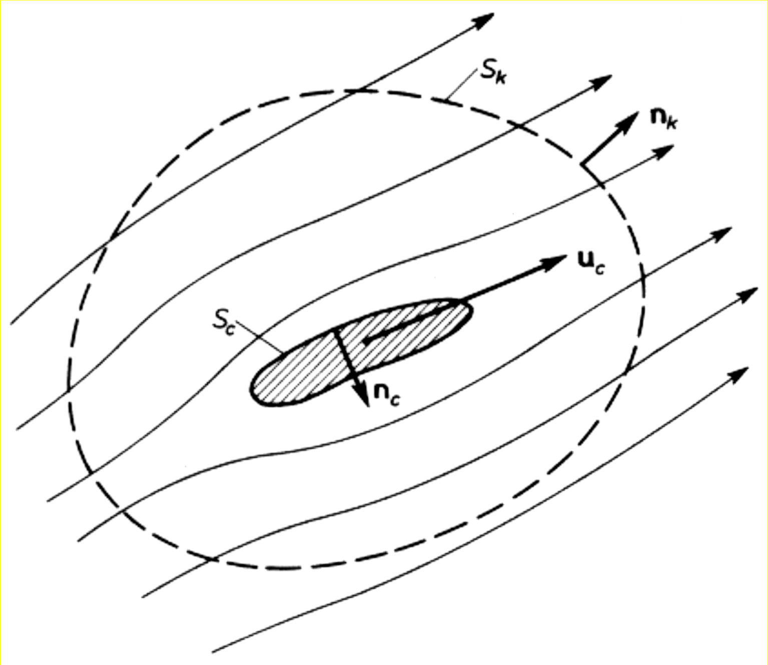
If the field of mass forces is a stationary potential field such that $\bar{f}=-grad\Pi$ then the energy conservation equation may be written as:

$$\rho \frac{D}{Dt}\left[\left(\frac{u^2}{2}+e+\Pi\right)\right]=div\left([P]\bar{u}\right)-div\bar{j}$$

or in the form of an integral equation:

$$\int_V \frac{\partial}{\partial t}\left[\rho\left(\frac{u^2}{2}+e+\Pi\right)\right]dV+\int_{S(V)}\rho\left(\frac{u^2}{2}+e+\Pi\right)u_n dS=\int_{S(V)}\bar{\tau}\bullet\bar{u}\,dS-\int_{S(V)}\bar{j}\bullet\bar{n}\,dS$$

Example No. 1: the case of an external flow around a body



S_k - the control surface

u_c - the velocity of the body

The energy conservation equation for such a case has the form:

$$\int_{V(S_k - S_c)} \frac{\partial}{\partial t}\left[\rho\left(\frac{u^2}{2}+e+\Pi\right)\right]dV + \int_{S_k} \rho\left(\frac{u^2}{2}+e+\Pi\right)u_n dS_k + \int_{S_c} \rho\left(\frac{u^2}{2}+e+\Pi\right)u_n dS_c =$$

$$= \int_{S_k} \bar{\tau}_{nn} \bullet \bar{u} dS_k + \int_{S_k} \bar{\tau}_{nt} \bullet \bar{u} dS_k + \int_{S_c} \bar{\tau}_{nn} \bullet \bar{u} dS_c + \int_{S_c} \bar{\tau}_{nt} \bullet \bar{u} dS_c - \int_{S_k} \bar{j} \bullet \bar{n}_k dS_k - \int_{S_c} \bar{j} \bullet \bar{n}_c dS_c$$

As the body surface is impermeable, the third term on the left hand side is equal zero. Moreover, if the velocity $\bar{u}_c$ is the same in all points of the body, we may write:

$$\int_{S_c} \bar{\tau}_{nn} \bullet \bar{u}_c dS_c + \int_{S_c} \bar{\tau}_{nn} \bullet \bar{u}_c dS_c = \bar{u}_c \left[\int_{S_c} \bar{\tau}_{nn} dS_c + \int_{S_c} \bar{\tau}_{nt} dS_c \right] = -\bar{u}_c \bullet \bar{R}$$

where *R* is the hydrodynamic reaction force acting on the body.

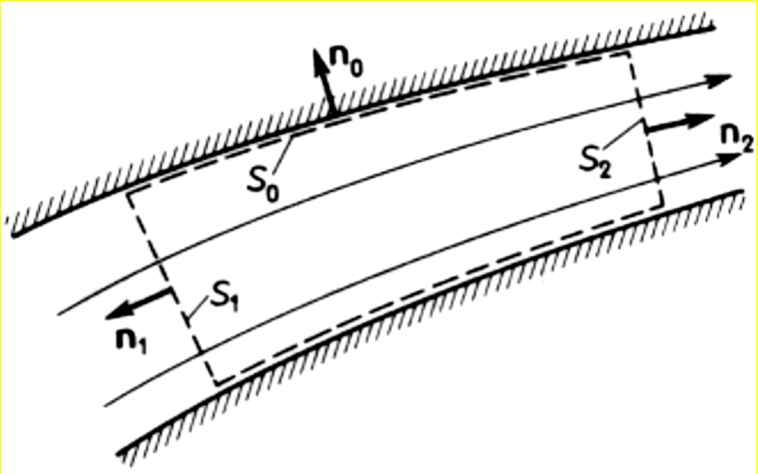
If we also assume: $\frac{\partial}{\partial t} = 0$ $\quad \bar{j} = 0$ $\quad \bar{\tau}_{nt} = 0$ $\quad \bar{\tau}_{nn} = -p\bar{n}$

we obtain:

$$\bar{u}_c \bullet \bar{R} = -\int_{S_k} \rho \left(\frac{u^2}{2} + e + \frac{p}{\rho} + \Pi \right) u_n dS_k$$

Conclusion: the hydrodynamic force acting on a body may be determined through the analysis of the hydrodynamic wake behind this body, without the direct measurement e.g. by means of a dynamometer.

Example No. 2: the case of internal flow (flow in a channel)



Assumption: the flow is stationary.

The energy conservation equation for such a case has the form:

$$m\left[\left(\frac{\tilde{u}^2}{2}+\tilde{e}+\tilde{\Pi}\right)_{S_2}-\left(\frac{\tilde{u}^2}{2}+\tilde{e}+\tilde{\Pi}\right)_{S_1}\right]=\int_{S_0}\bar{\tau}_{nn}\bullet\bar{u}\,dS_0+\int_{S_0}\bar{\tau}_{nt}\bullet\bar{u}\,dS_0+\int_{S_1}\bar{\tau}_{nn}\bullet\bar{u}\,dS_1+$$

$$+\int_{S_1}\bar{\tau}_{nt}\bullet\bar{u}\,dS_1+\int_{S_2}\bar{\tau}_{nn}\bullet\bar{u}\,dS_2+\int_{S_2}\bar{\tau}_{nt}\bullet\bar{u}\,dS_2-\int_{S_0}\bar{j}\bullet\bar{n}_0\,dS_0-\int_{S_1}\bar{j}\bullet\bar{n}_1\,dS_1-\int_{S_2}\bar{j}\bullet\bar{n}_2\,dS_2$$

m – mass intensity of flow

The left hand side of the equation contains mean values defined in the following way:

$$\left(\frac{\tilde{u}^2}{2}+\tilde{e}+\tilde{\Pi}\right)_S=\frac{1}{m}\int_S \rho\left(\frac{u^2}{2}+e+\Pi\right)u_n dS$$

Additional assumptions:

-the fluid is inviscid, so the tangential stresses are zero and the normal stresses are expressed only by pressure,

-There is no heat transport through the inlet and outlet surfaces.

This leads to the expression: where: $\Pi = gz$

$$m\left[\left(\frac{\tilde{u}^2}{2}+\tilde{e}+\frac{\tilde{p}}{\tilde{\rho}}+\tilde{\Pi}\right)_{S_1}-\left(\frac{\tilde{u}^2}{2}+\tilde{e}+\frac{\tilde{p}}{\tilde{\rho}}+\tilde{\Pi}\right)_{S_2}\right]=-\int_{S_0}\bar{j}\bullet\bar{n}_0 dS_0$$

This expression may describe for example the flow through the pipe in a heat exchanger.