J. Szantyr – Lecture No. 14 – The closed system of equations of the fluid mechanics

The above presented equations form the closed system of the fluid mechanics equations, which may be employed for description of realistic flows and for obtaining, through solution of these equations, information about the values of the interesting parameters describing these flows. The actual format of the system of equations depends on the adopted fluid model and flow model.

Case No. 1: Incompressible fluid of constant viscosity

The closed system of equations is formed of:

- mass conservation equation $div\overline{u} = 0$

- momentum conservation equation $\rho \frac{D\overline{u}}{Dt} = \rho\overline{f} - gradp + \mu\Delta\overline{u}$

These are equivalent to four scalar equations with four unknowns:

- pressure p

- velocity components u_x, u_y, u_z

In this case the temperature field does not influence the flow, but it depends itself on the velocity field of the flow through the entropy balance equation in the form:

$$\rho c\left(\frac{\partial T}{\partial t}+u_x\frac{\partial T}{\partial x}+u_y\frac{\partial T}{\partial y}+u_z\frac{\partial T}{\partial z}\right)=T\dot{s}_M+\lambda\Delta T$$

This form of the equation may be obtained from the original formula by substituting the relation for the fluid internal energy: $e = cT + e_0$

In the case when the fluid viscosity depends on the temperature, the balance of entropy equation is connected with the mass and momentum conservation equations through the relation: $\mu = \mu(T)$

Then we have the system of six equations with six unknowns:

- pressure p
- velocity components u_x, u_y, u_z
- temperature T
- viscosity coefficient μ

Case No. 2: compressible fluid

In this case the closed system of equations is formed of:

- mass conservation equation $$\frac{\partial \rho}{\partial t} + div\left(\rho \bar{u}\right) = 0$$

- momentum conservation eq. $$\rho \frac{D\bar{u}}{Dt} = \rho \bar{f} - grad p - grad\left(\frac{2}{3}\mu div \bar{u}\right) + div\left(2\mu\left[D\right]\right)$$

- entropy balance equation $$\rho \frac{De}{Dt} = T\dot{s}_M + \frac{p}{\rho}\frac{Dp}{Dt} + \lambda \Delta T$$

- internal energy equation $$e = \int_{T_0}^{T} c_v\left(T\right) dT$$

- equation of state $$\frac{p}{\rho} = Z\left(p, T\right) RT$$

Z – compressibility function

R – gas constant

- additional relations $$\mu = \mu\left(T\right) \qquad c_V = c_V\left(T\right)$$

In this case we have the system of nine equations with nine unknowns:

- pressure p
- density ρ
- internal energy e
- temperature T
- viscosity coefficient μ
- velocity components u_x, u_y, u_z
- specific heat c_V

It was assumed that the thermal conductivity coefficient λ is constant and given.

Boundary and initial conditions

In order to enable solution of the above systems of equations it is necessary to determinet he appropriate boundary and (for unsteady flows) initial conditions. These conditions are required to determine the arbitrary constants and arbitrary functions resulting from the integration of the equations.

Boundary conditions

Boundary conditions on the surface of an impermeable body

- inviscid fluid – normal velocity equal zero $u_n = 0$
- viscous fluid – total velocity equal zero $\overline{u} = 0$
- given temperature T or stream of heat j

Boundary conditions on the surface of a porous body

- tangential velocity equal zero $u_t = 0$
- normal velocity given $u_n = f\left(x, y, z\right)$
- given temperature T or stream of heat j

Boundary conditions on the free surface between two fluids

- inviscid fluid $u_n^1 = u_n^2$
- viscous fluid $\overline{u}_1 = \overline{u}_2$

If the free surface equation between two fluids is described by:

$$F(x, y, z, t) = 0$$

Then the kinematic boundary condition has the form:

$$\frac{\partial F}{\partial t} + u_x \frac{\partial F}{\partial x} + u_y \frac{\partial F}{\partial y} + u_z \frac{\partial F}{\partial z} = 0$$

<u>Boundary conditions at infinity</u> are given in the case of flow around an object in which the velocity field far from the object is uniform.

$$\bar{u} = \bar{u}(\infty) \qquad p = p(\infty) \qquad T = T(\infty)$$

Initial conditions

The initial conditions refer to the unsteady flow phenomena and they should be determined for every point of space filled with the fluid for the initial instant of time $t = t_0$

For the incompressible fluid flow:

$$p = p(x, y, z, t_0) \qquad \overline{u} = \overline{u}(x, y, z, t_0)$$

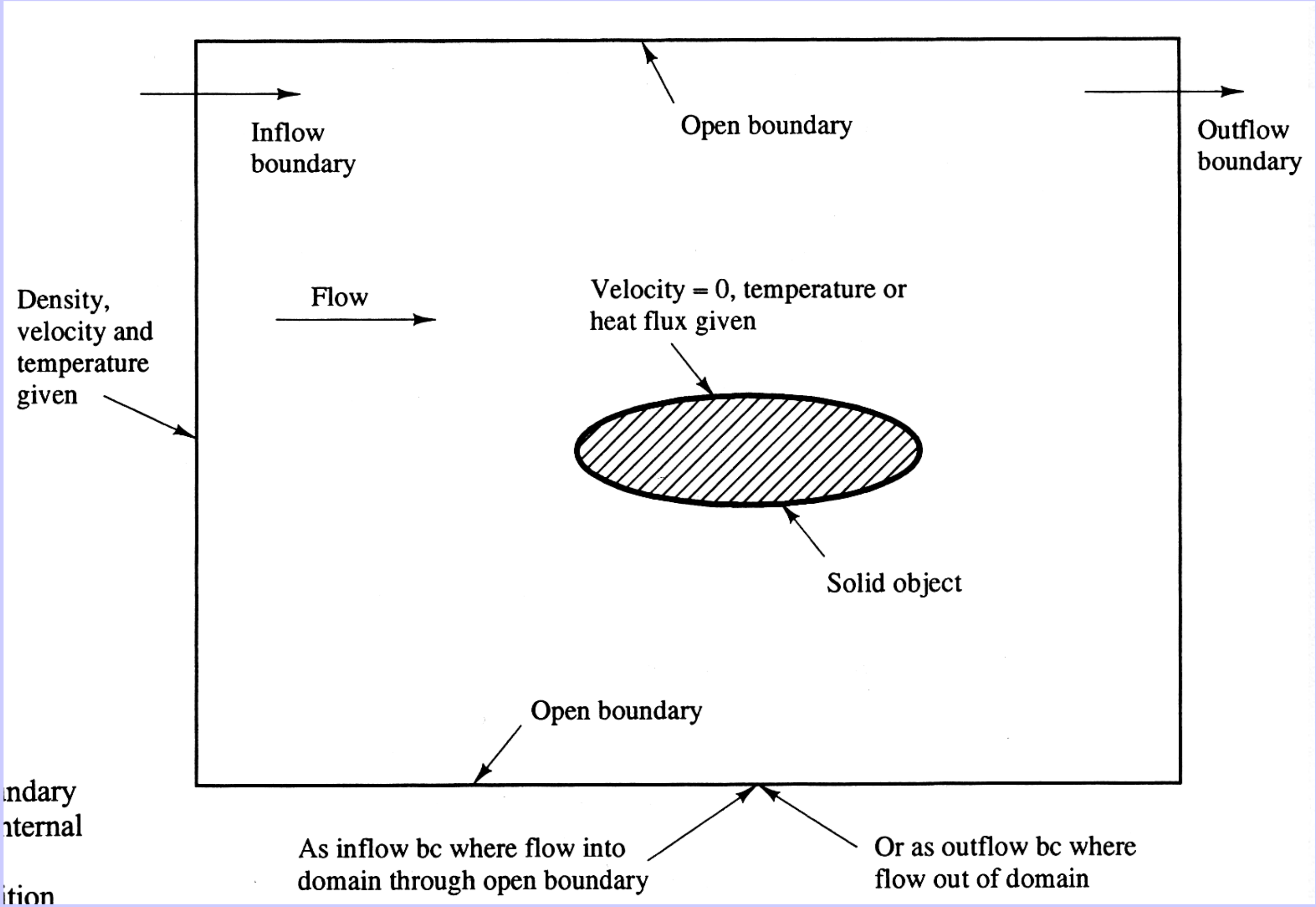
For the incompressible fluid flow with viscosity dependent on temperature, additionally:

$$T = T(x, y, z, t_0) \qquad \mu = \mu(x, y, z, t_0)$$

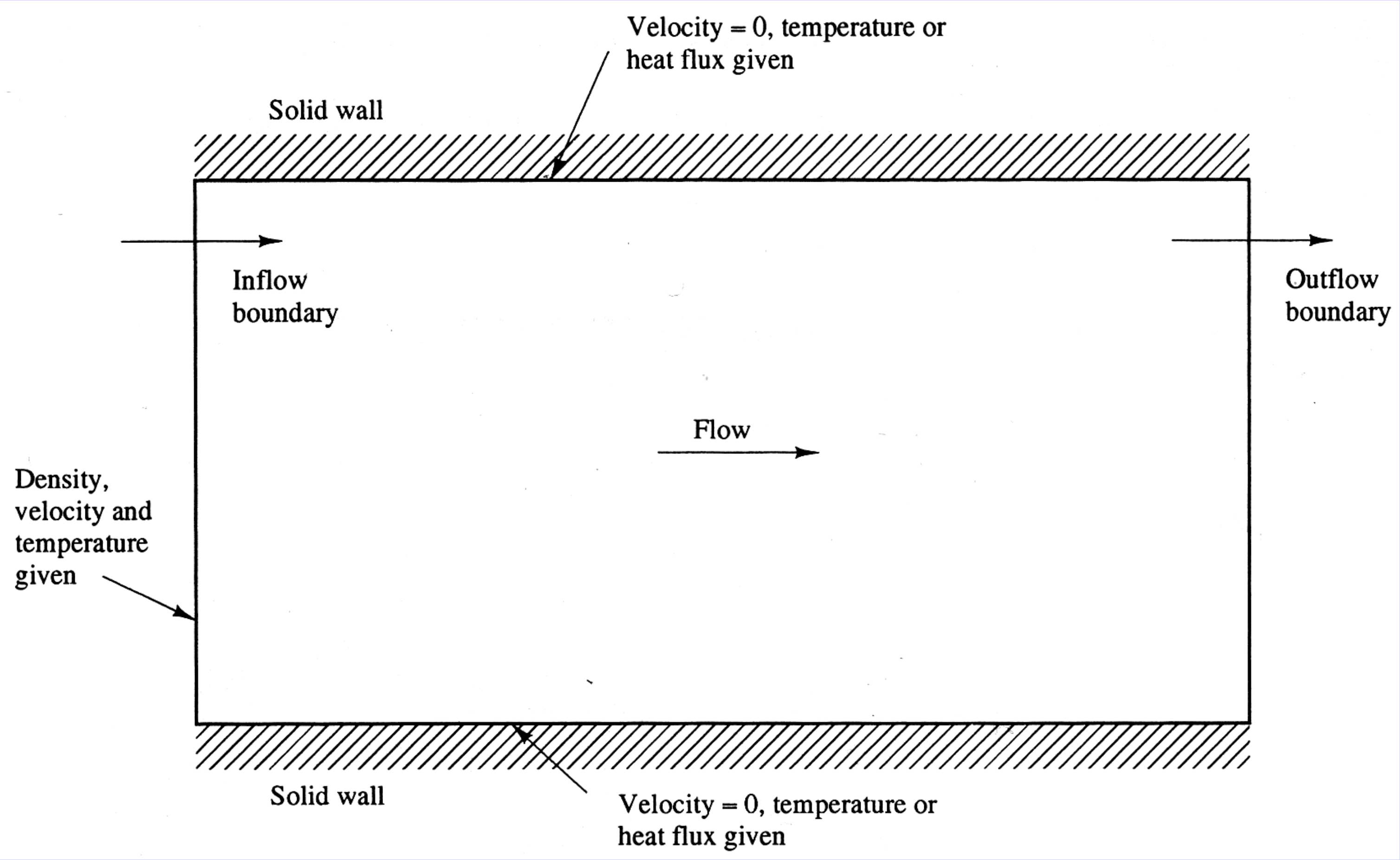
For the compressible fluid flow, additionally:

$$\rho = \rho(x, y, z, t_0) \qquad e = e(x, y, z, t_0) \qquad c_V = c_V(x, y, z, t_0)$$

The initial conditions should not be in conflict with the boundary conditions.



Scheme of the boundary conditions for an external flow



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