J. Szantyr – Lecture No. 15 – Bernoulli equation

Bernoulli equation expresses, under certain assumptions, the principles of momentum conservation and energy conservation of the fluid.

Assumptions:

-the flow is stationary $\frac{\partial}{\partial t} = 0$

-the fluid is inviscid $\mu = 0$

-the fluid is barotropic $\rho = \rho(p)$

-the mass forces form a potential field $\bar{f} = -grad\Pi$

Under such assumptions the Euler equation may be integrated:

$$\rho \frac{D\bar{u}}{Dt} = \rho\bar{f} - gradp$$

We use the identity:

$$\frac{D\overline{u}}{Dt} = \frac{\partial \overline{u}}{\partial t} + \overline{u} grad \overline{u} = \frac{\partial \overline{u}}{\partial t} + grad \frac{u^2}{2} + rot \overline{u} \times \overline{u}$$

Moreover, we introduce a pressure function: $P(p) = \int_{p_0}^{p} \frac{dp}{\rho(p)}$

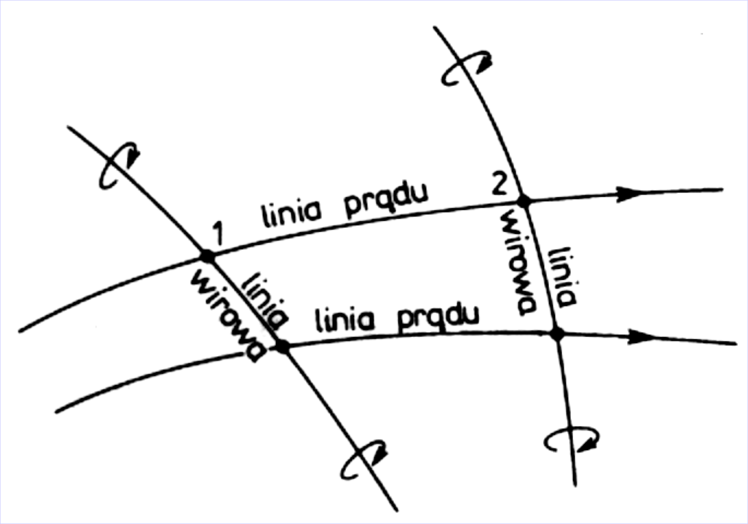
what leads to: $\frac{1}{\rho} grad p = grad P(p)$

Then the Euler equation may be written as:

$$\frac{\partial \overline{u}}{\partial t} + grad\left[\frac{u^2}{2} + P(p) + \Pi\right] = rot \overline{u} \times \overline{u}$$

or: $$grad\left[\frac{u^2}{2} + P(p) + \Pi\right] = grad E = rot \overline{u} \times \overline{u}$$

The expression in brackets is known as the Bernoulli tri-nomial *E*. It may be proved that *E* is constant in five cases.



Case 1: along a stream-line

$$gradE = \bar{u} \times rot\bar{u}$$

Both sides are multipled by the unit length vector along the stream-line $\bar{i}_s$

$$gradE \bullet \bar{i}_s = (\bar{u} \times rot\bar{u}) \bullet \bar{i}_s = 0$$

This leads to: $\frac{dE}{ds} = 0$ or: *E=const*

Case 2: along a vortex line

Both sides are multiplied by the unit length vector along the vortex line

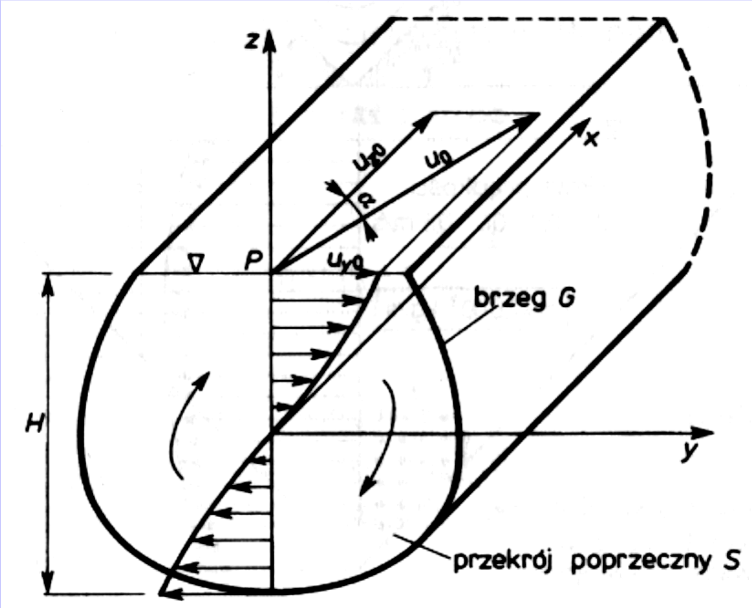
$gradE \bullet \bar{i}_\omega = (\bar{u} \times rot\bar{u}) \bullet \bar{i}_\omega = 0$ this gives $\frac{dE}{d\omega} = 0$ or *E=const*

Stream-lines and vortex lines form so called Bernoulli surface, at which there is *E=const.*

Case 3: an irrotational flow $rot\bar{u} = 0 \rightarrow E = const$

Case 4: a hydrostatic situation $\bar{u} = 0 \rightarrow E = const$

Case 5: a helicoidal flow



$$rot\bar{u} = \lambda\bar{u}$$

$$\bar{u} \times rot\bar{u} = \bar{u} \times \lambda\bar{u} = 0 \rightarrow E = const$$

In the case of an **incompressible fluid** flow in the **gravitational field** we have: $\rho = const$ and $\Pi = gz$ what gives:

Other forms of the Bernoulli equation are possible if particular forms of the barotropic relation are adopted. For example, in case of a gas undergoing an adiabatic process this relation reads:

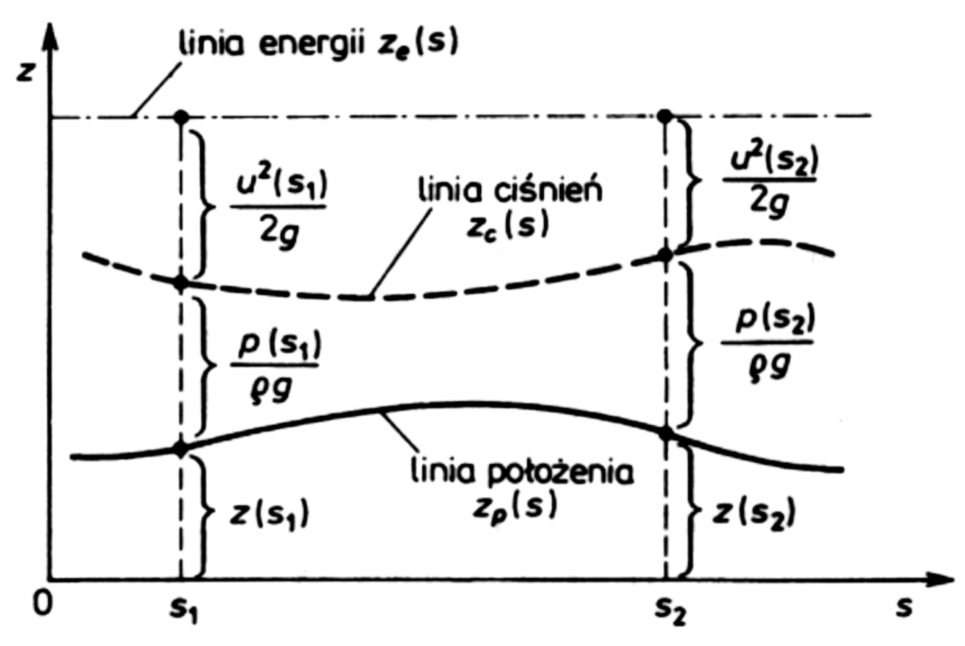
$$\rho = \frac{\rho_0}{p_0^{1/\kappa}} p^{1/\kappa}$$ where κ jest is the Poisson adiabatic exponent $\kappa = \frac{c_p}{c_v}$

Then the Bernoulli equation takes the form:

$$\frac{u^2}{2} + \frac{\kappa}{\kappa - 1} \frac{p_0}{\rho_0} \left[\left(\frac{p}{p_0} \right)^{(\kappa-1)/\kappa} - 1 \right] + gz = const$$

Comparison of the Bernoulli equation development with the energy conservation equation for a stream tube (Lecture No 12, slide 7) shows that, with disregarding the fluid internal energy e and the thermal conductivity of the fluid, the Bernoulli equation describes the energy conservation principle as well.

The Bernoulli Equation (1738)



$$gz + \frac{p}{\rho} + \frac{u^2}{2} = const$$

or

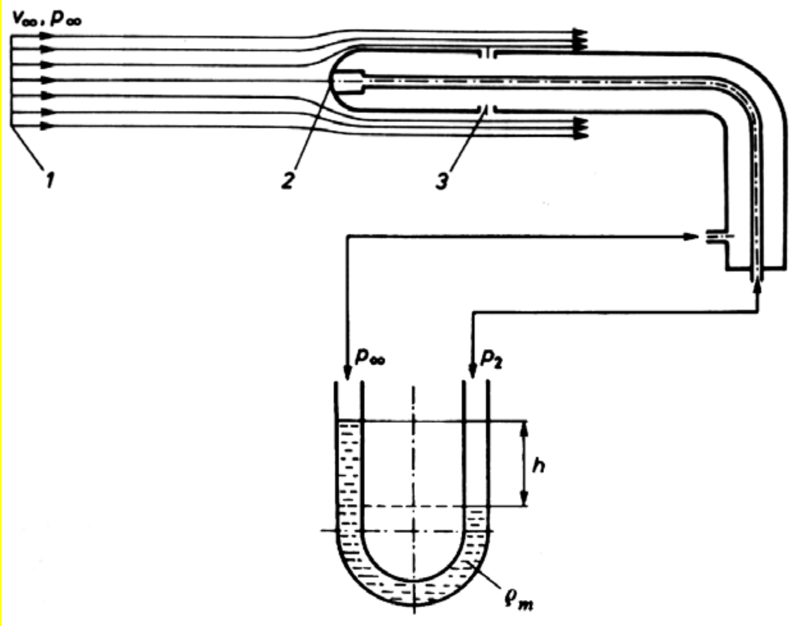
$$z + \frac{p}{\rho g} + \frac{u^2}{2g} = const$$

The sum of the potential energy of the mass forces, the pressure energy and the kinetic energy of the fluid is constant.

or:

The sum of the geometrical elevation, the pressure head (i.e. the height at the fluid is elevated under pressure *p*) and the velocity head (i.e. the height from which the falling fluid achieves velocity *u*) is constant.

Example No. 1



What is the velocity of a fluid flow measured by the Prandtl tube, if the connected to it manometer shows the level difference h of the manometric fluid of density ρ_m ?

The Bernoulli equation for the points 2 and 3 has the form:

$$\frac{u_2^2}{2} + \frac{p_2}{\rho} = \frac{u_3^2}{2} + \frac{p_3}{\rho}$$

where: $u_2 = 0$ $u_3 = v_\infty$ $p_3 = p_\infty$

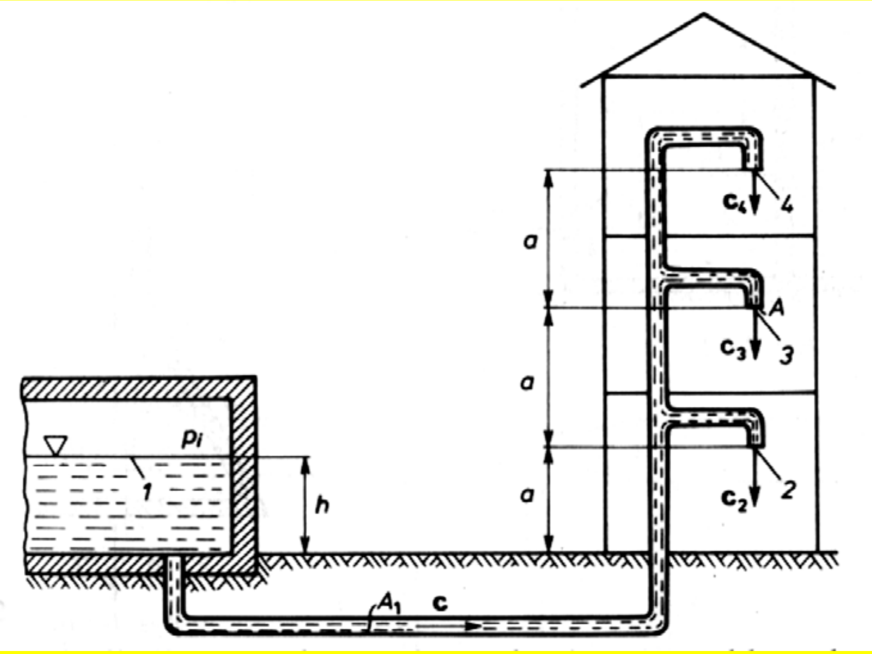
What leads to: $$\frac{v_\infty^2}{2} = \frac{p_2 - p_\infty}{\rho}$$

In turn the pressure difference on the manometer is:

$p_2 - p_\infty = \rho_m g h$ or: $\frac{v_\infty^2}{2} = \frac{\rho_m}{\rho} g h$ and finally:

$$v_\infty = \sqrt{2 \frac{\rho_m}{\rho} g h}$$

Example No. 2



A building is supplied with water from a large tank in which the absolute pressure is p_i and the distance of the water level from the foundation is *h=const*. Determine the velocity of flow on each floor and the velocity of flow in the underground pipe of cross-section A_1

Given: all cross-sections of pipes in the building are equal *A*, the density of water is equal *ρ*, and the barometric pressure is equal p_b

The Bernoulli equations with respect to the foundation level are:

$$\frac{c_1^2}{2}+\frac{p_i}{\rho}+gh=\frac{c_2^2}{2}+\frac{p_b}{\rho}+ga \qquad \frac{c_1^2}{2}+\frac{p_i}{\rho}=\frac{c_3^2}{2}+\frac{p_b}{\rho}+2ga$$

$$\frac{c_1^2}{2}+\frac{p_i}{\rho}=\frac{c_3^2}{2}+\frac{p_b}{\rho}+3ga$$

The velocity of the water level in the tank is equal zero, hence we have:

$$c_1 = \sqrt{2\left[\frac{p_i - p_b}{\rho} + g(h-a)\right]}$$

$$c_2 = \sqrt{2\left[\frac{p_i - p_b}{\rho} + g(h-2a)\right]}$$

$$c_3 = \sqrt{2\left[\frac{p_i - p_b}{\rho} + g(h-3a)\right]}$$

The velocity c in the underground tube may be dermined from the continuity condition:

$$cA_1 = c_2 A + c_3 A + c_4 A$$

what gives:

$$c = \frac{A}{A_1}\left(c_2 + c_3 + c_4\right)$$