J. Szantyr – Lecture No. 16 – Similarity of flows I

Experimental testing of flows is most frequently performed on models, which are reduced copies of real objects, manufactured at a given scale. In order to produce valid experimental results for the real object, certain similarity conditions between the model and real flows must be observed.

System of measuring of the physical quantities

Basic units		Derived units	
Length	[m]	Force	$[N]=\left[kg\frac{m}{s^2}\right]$
Mass	[kg]	Power	$[W]=\left[kg\frac{m^2}{s^3}\right]$
Time	[s]		
Temperature	[K]		

Formulation of the laws of physics does not depend on the units.

The Buckingham theorem (Π theorem)

1. Every function of *n* dimensional parameters, of which *k* have basic units of measure, may be expressed as the function of *n*-*k* non-dimensional parameters of the form:

$$\Pi_{k+1} = \frac{a_{k+1}}{a_1^{p_1} a_2^{p_2} .....a_k^{p_k}}$$

2. If the non-dimensional parameters Π are identical for two different situations (e.g. in two different scales), the phenomena are identical in both situations, despite the differences in the dimensional parameters *a*. Consequently, the parameters Π may be regarded as **similarity parameters** or **criteria of similarity**.

On the basis of the above theorem a dimensional analysis of the fluid mechanics equations may be performed and the corresponding criteria of similarity may be developed.

Dimensional analysis of the mass conservation equation

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u_x)}{\partial x}+\frac{\partial(\rho u_y)}{\partial y}+\frac{\partial(\rho u_z)}{\partial z}=0$$ for scale 1

$$\frac{\partial \rho'}{\partial t'}+\frac{\partial(\rho' u_x')}{\partial x'}+\frac{\partial(\rho' u_y')}{\partial y'}+\frac{\partial(\rho' u_z')}{\partial z'}=0$$ for scale 2

We introduce scale coefficients: $\rho' = \alpha_\rho \rho$ $\quad t' = \alpha_t t$

$x' = \alpha_x x \quad y' = \alpha_y y \quad z' = \alpha_z z \quad u_x' = \alpha_{ux} u_x \quad u_y' = \alpha_{uy} u_y \quad u_z' = \alpha_{uz} u_z$

We assume geometrical similarity of flows in both scales, that is:

$$\alpha_x = \alpha_y = \alpha_z = \alpha_l = \frac{l'}{l}$$

Moreover, we assume kinematical similarity, i.e. the similarity of the velocity fields in flows in both scales, that is:

$$\alpha_{ux} = \alpha_{uy} = \alpha_{uz} = \alpha_u = \frac{\bar{u}'}{\bar{u}}$$

Now the equation in scale 2 may be written in the form:

$$\frac{\alpha_\rho}{\alpha_t}\frac{\partial\rho}{\partial t}+\frac{\alpha_\rho\alpha_u}{\alpha_l}\left[\frac{\partial(\rho u_x)}{\partial x}+\frac{\partial(\rho u_y)}{\partial y}+\frac{\partial(\rho u_z)}{\partial z}\right]=0$$

The condition for identity of the equation in scales 1 and 2 is:

$$\frac{\alpha_\rho}{\alpha_t}=\frac{\alpha_\rho\alpha_u}{\alpha_l} \qquad \text{or} \qquad \frac{\alpha_l}{\alpha_t\alpha_u}=1$$

Hence we have the equation:

$$\frac{l}{tu}=\frac{l'}{t'u'}=\frac{t_c}{t}=Sh$$

Sh – Strouhal number

t_c - the characteristic time of flow (i.e. the time, in which the fluid covers the characteristic distance *l* – e.g. pipe length, with the characteristic velocity *u*)

t - the time of variation of the unsteady flow parameters, e.g. the time cycle of the piston pump operation

Using the Strouhal number, the mass conservation equation may be written in a non-dimensional form:

$$\boxed{Sh\frac{\partial\hat{\rho}}{\partial t}+\frac{\partial\left(\hat{\rho}\hat{u}_x\right)}{\partial\hat{x}}+\frac{\partial\left(\hat{\rho}\hat{u}_y\right)}{\partial\hat{y}}+\frac{\partial\left(\hat{\rho}\hat{u}_z\right)}{\partial\hat{z}}=0}$$

where all quatities are related to the corresponding characteristic quantities, what makes them non-dimensional, for example:

$$\hat{\rho}=\frac{\rho}{\rho_0}\qquad \hat{t}=\frac{t}{t_0}\qquad \hat{u}_x=\frac{u_x}{u_0}\qquad \hat{x}=\frac{x}{x_0}\qquad \text{etc.}$$

A small value of the Strouhal number in a given flow means that the non-stationary phenomena in this flow are meaningless and they may be neglected.

Dimensional analysis of the Navier – Stokes equation

Additionally, the following scale coefficient must be introduced:

$$\bar{f}' = \alpha_f \bar{f} \qquad \mu' = \alpha_\mu \mu \qquad p' = \alpha_p p$$

After substitution to the N-S equation we obtain:

$$\frac{\alpha_\rho \alpha_u}{\alpha_t} \rho \frac{\partial \bar{u}}{\partial t} + \frac{\alpha_\rho \alpha_u^2}{\alpha_l} \rho \left[u_x \frac{\partial \bar{u}}{\partial x} + u_y \frac{\partial \bar{u}}{\partial y} + u_z \frac{\partial \bar{u}}{\partial z} \right] = \alpha_\rho \alpha_f \rho \bar{f} - \frac{\alpha_p}{\alpha_l} grad p +$$

$$- \frac{\alpha_\mu \alpha_u}{\alpha_l^2} grad\left(\frac{2}{3} \mu div \bar{u} \right) + \frac{\alpha_\mu \alpha_u}{\alpha_l^2} div(2\mu[D])$$

This equation is identical in two different scales 1 and 2 under the following condition:

$$\frac{\alpha_\rho \alpha_u}{\alpha_t} = \frac{\alpha_\rho \alpha_u^2}{\alpha_l} = \alpha_\rho \alpha_f = \frac{\alpha_p}{\alpha_l} = \frac{\alpha_\mu \alpha_u}{\alpha_l^2}$$

After dividing both sides by the second term and using the scale coefficients we get:

Strouhal number: $$Sh = \frac{l}{tu} = \frac{l'}{t'u'}$$

Froude number: $$(Fr)^2 = \frac{u^2}{fl} = \frac{u'^2}{f'l'}$$

Froude number defines the ratio of inertia forces and mass forces

Euler number: $$Eu = \frac{p}{\rho u^2} = \frac{p'}{\rho' u'^2}$$

Euler number defines the ratio of pressure forces to inertia forces

Reynolds number: $$\mathrm{Re} = \frac{\rho u l}{\mu} = \frac{\rho' u' l'}{\mu'}$$

Reynolds number defines the ratio of inertia forces to viscosity forces

Using the Strouhal, Froude, Eulera and Reynolds numbers the Navier – Stokes equation may be written in a non-dimensional form:

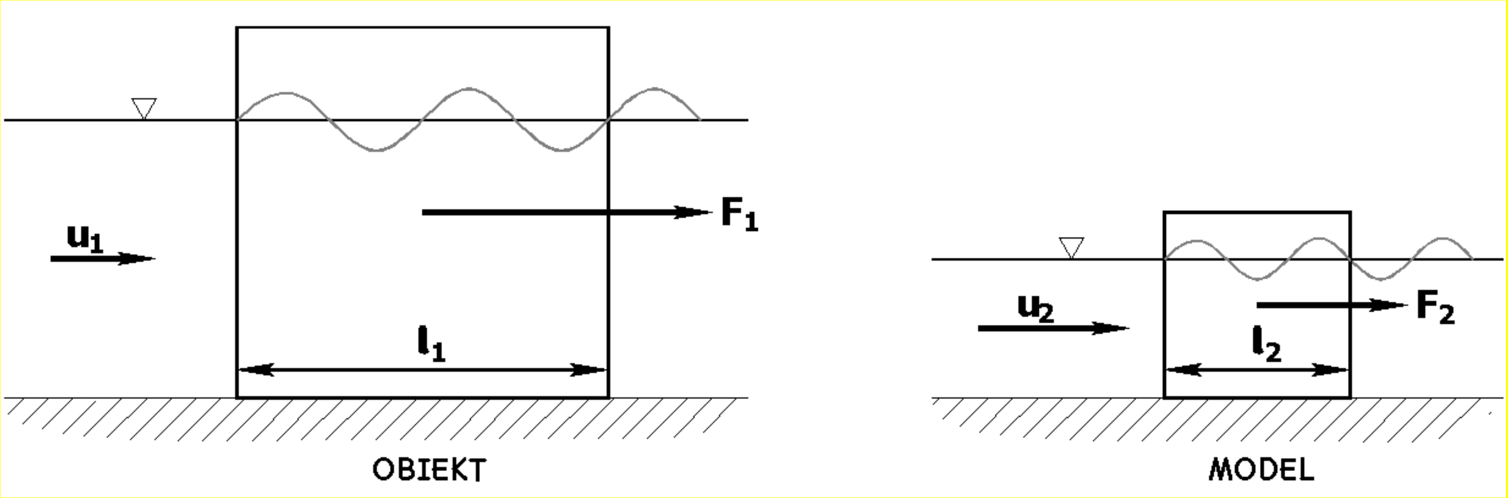
$$Sh\rho\frac{\partial\bar{u}}{\partial t}+\rho(\bar{u}grad)\bar{u}=\frac{\rho}{(Fr)}\bar{f}-Eu\cdot gradp+$$

$$-\frac{1}{\mathrm{Re}}\left[grad\left(\frac{2}{3}\mu div\bar{u}\right)-div(2\mu[D])\right]$$

In the above equation all parameters are related to their characteristic values, similarly as in the mass conservation equation.

When the N-S equation in the above form is applied to flows in two different scales, we obtain full similarity of the phenomena if <u>all above similarity criteria are fulfilled</u>. This is not always possible. If only some of the criteria are fullfilled, we obtain only <u>partial similarity</u>, and the results of measurements or calculations are subject to so called scale effects (see the example below).

Example



We consider the case of a bridge support (1), which is tested as a model (2) in the reduced scale (1:10), in order to determine the resultant hydrodynamic force F acting on the support. This force includes a viscous contribution, dependent mainly on the Reynolds number, and a wave contribution, dependent mainly on the Froude number. We assume:

$$F = F_w(Fr) + F_v(\mathrm{Re}) \qquad \text{at e.g.:} \qquad U_1 = 5{,}0\left[\frac{m}{s}\right]$$

In case of full similarity $\mathrm{Re}_1 = \mathrm{Re}_2 \quad Fr_1 = Fr_2$ it may be written:

$$F_1 = C_{F1}\frac{\rho_1}{2}U_1^2 S_1 \quad \text{where:} \quad C_{F1} = C_{F2} = \frac{F_2}{\frac{\rho}{2}U_2^2 S_2}$$

Equality of the Froude numbers leads to:

$$\frac{U_1}{\sqrt{gL_1}} = \frac{U_2}{\sqrt{gL_2}} \rightarrow U_2 = U_1\sqrt{\frac{L_2}{L_1}} = 0{,}3162 \rightarrow U_2 = 1{,}581\left[\frac{m}{s}\right]$$

In turn, equality of the Reynolds numbers leads to:

$$\frac{U_1 L_1}{\nu} = \frac{U_2 L_2}{\nu} \rightarrow U_1 L_1 = U_2 L_2 \rightarrow U_2 = U_1\frac{L_1}{L_2} \rightarrow U_2 = 50{,}0\left[\frac{m}{s}\right]$$

It is visible that both criteria cannot be fulfilled simultaneously. It is easier to fulfil the Froude criterion, because the Reynolds criterion requires application of a very high velocity in the laboratory. This leads to the scale effect, which should be taken into account when the model results are re-calculated into full scale.

This leads to the formula: $$F_1 = \left(C_{W1} + C_{V1}\right)\frac{\rho}{2}U_1^2 S_1$$

where: $$C_{W1} = C_{W2} = C_{F2} - C_{V2}$$

$$C_{F2} = \frac{F_2}{\frac{\rho}{2}U_2^2 S_2}$$

$$C_{V2} = f\left(\mathrm{Re}_2\right)$$

$$C_{V1} = f\left(\mathrm{Re}_1\right)$$

The scale effect is included in: $$\Delta C_V = C_{V1} - C_{V2}$$

because there is: $$\mathrm{Re}_1 \neq \mathrm{Re}_2$$