J. Szantyr – Lecture No. 23 – Potential flows 1

If the fluid flow is irrotational, i.e. everywhere or almost everywhere in the field of flow there is $rot\overline{u} = 0$ it means that there exists a scalar function $\varphi(x, y, z)$, such that $\overline{u} = grad\varphi$. Such a flow is called the potential flow, and function φ is called the velocity potential.

We have: $u_x = \frac{\partial\varphi}{\partial x}$ $\quad u_y = \frac{\partial\varphi}{\partial y}$ $\quad u_z = \frac{\partial\varphi}{\partial z}$

In case of potential flow of an incompressible fluid the mass conservation equation is transformed into the Laplace equation:

$$\frac{\partial\rho}{\partial t} + div(\rho\overline{u}) = 0 \rightarrow divgrad\varphi = \Delta\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0$$

The Laplace equation is linear, that means that the sum of its solutions is also a solution. In practise, very complicated potential functions, describing complex flows, may be composed of the simple functions describing so called elementary flows.

The potential flows are particularly suited for mathematical modelling of the fluid motion in the regions outside the boundary layers and wakes, where the influence of the fluid viscosity is negligibly small. The method of formation of the complicated potential flows will be demonstrated using the example of two-dimensional flows.

In this case we have:

$$u_x = \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad u_y = \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

where:

$\varphi(x, y)$ - the velocity potential

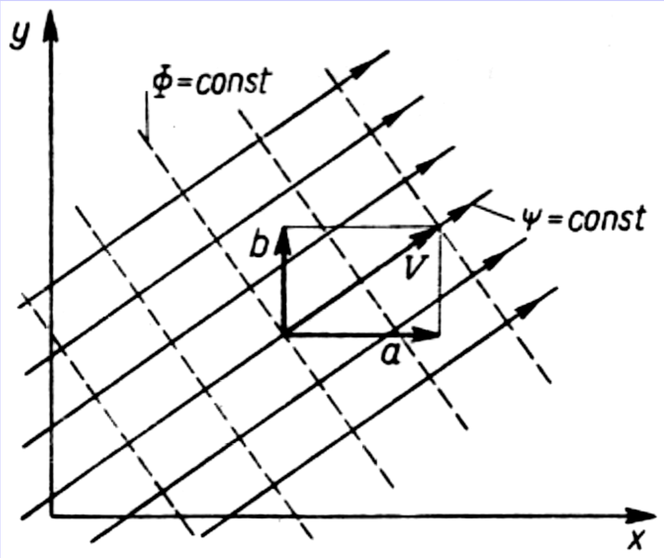
$\varphi(x, y) = C$ - the equipotential lines

$\psi(x, y)$ - the stream function

$\psi(x, y) = C$ - the stream lines

The elementary potential flows

1. The uniform flow



$$u_x = a = \frac{\partial \varphi}{\partial x} \qquad u_y = b = \frac{\partial \varphi}{\partial y}$$

The velocity potential:

$$\varphi(x, y) = a \cdot x + b \cdot y = u_x \cdot x + u_y \cdot y$$

The equipotential lines:

$$y = -\frac{a}{b} x + C = -\frac{u_x}{u_y} x + C$$

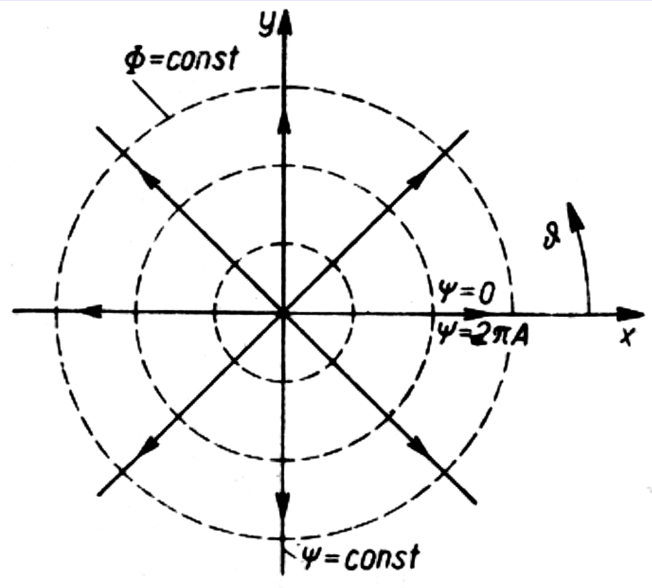
The stream function:

$$\psi(x, y) = a \cdot y - b \cdot x = u_x \cdot y - u_y \cdot x$$

The stream lines:

$$y = \frac{b}{a} x + C = \frac{u_y}{u_x} x + C$$

2. The source (positive or negative)



The source is a singular point in the field of flow, in which an outflow of fluid of a certain volumetric intensity Q takes place. This outflow is uniform in all directions. In case of a negative source (or a sink), the fluid flows into the sink and „disappears”. Thus we have:

$Q = \pm 2\pi r u_r$ or: $u_r = \pm \dfrac{Q}{2\pi r}$ where: u_r - the radial velocity

$$u_r = \frac{\partial \varphi}{\partial r} = \pm \frac{Q}{2\pi r} \rightarrow \varphi = \pm \frac{Q}{2\pi} \ln r$$

The constant values of the potential φ correspond to the constant values of the radius r, i.e. the equipotential lines are circles having a common centre.

In the Cartesian system of co-ordinates we have:

$$u_x = u_r \cos\theta = \frac{Q}{2\pi}\frac{x}{\left(x^2 + y^2\right)} \qquad u_y = u_r \sin\theta = \frac{Q}{2\pi}\frac{y}{\left(x^2 + y^2\right)}$$

where: $\theta = arctg\,\frac{y}{x}$

and further we have:

the complete differential of the potential:

$$d\varphi = \frac{Q}{2\pi}\frac{xdx}{\left(x^2 + y^2\right)} + \frac{Q}{2\pi}\frac{ydy}{\left(x^2 + y^2\right)}$$

the complete differential of the stream function:

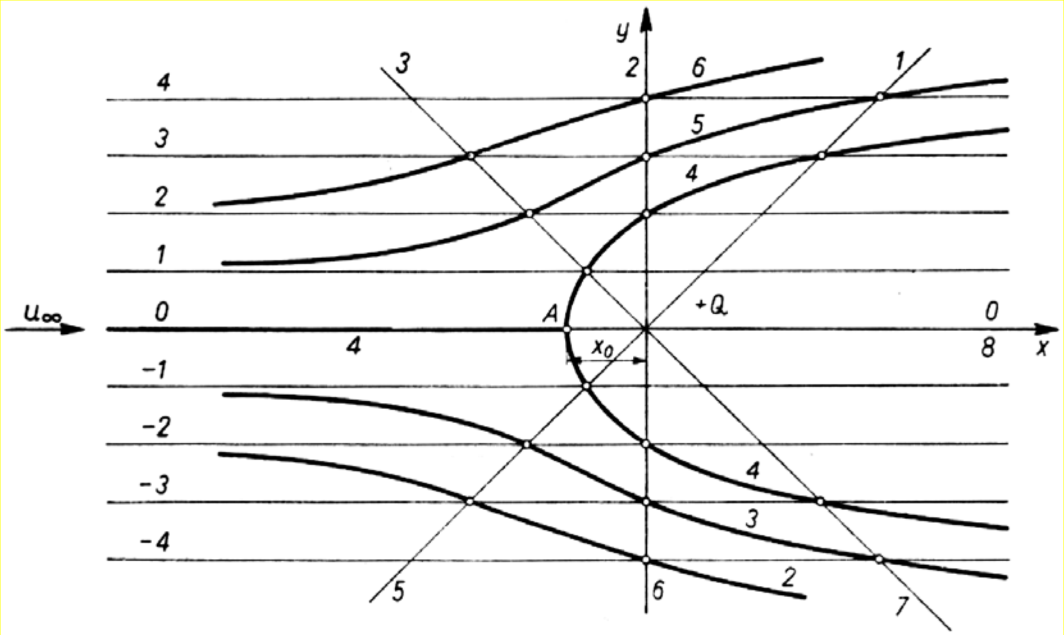
$$d\psi = \frac{Q}{2\pi}\frac{ydx}{\left(x^2 + y^2\right)} - \frac{Q}{2\pi}\frac{xdy}{\left(x^2 + y^2\right)}$$

and further, after integration we have:

$$\varphi = \frac{1}{2}\frac{Q}{2\pi}\ln\left(x^2 + y^2\right) \qquad \psi = \frac{Q}{2\pi}arctg\,\frac{y}{x}$$

Stream lines are straight lines passing through the source

Example: Superposition of an uniform flow and a source



The potential and stream function of the resultant flow are the sums of potentials and stream functions We assume that the uniform flow is parallel to the x axis.

The potential: $$\varphi = u_\infty x + \frac{Q}{4\pi}\ln\left(x^2 + y^2\right)$$

The stream function: $$\psi = u_\infty y + \frac{Q}{2\pi} arctg\frac{y}{x}$$

The zero-th stream line: $$\psi = 0 \rightarrow u_\infty y + arctg\frac{y}{x} = 0$$

Solution for the zero-th stream line: $x = -y\,ctg\left(\frac{2\pi u_\infty}{Q}\right)$

If we replace the zero-th stream line with a solid wall, the picture of flow will not change and we obtain the flow around a solid „half-body".

The velocity components:

$$u_x = u_\infty + \frac{Q}{2\pi}\frac{x}{\left(x^2+y^2\right)} \qquad u_y = \frac{Q}{2\pi}\frac{y}{\left(x^2+y^2\right)}$$

The location of the stagnation point on the x axis: $x_0 = -\frac{Q}{2\pi u_\infty}$

The pressure *p* in an arbitrary point of the flow may be calculated according to the Bernoulli equation:

$$p_\infty + \frac{\rho u_\infty^2}{2} = p + \frac{\rho u^2}{2} \rightarrow C_p = \frac{p - p_\infty}{\frac{1}{2}\rho u_\infty^2} = 1 - \left(\frac{u}{u_\infty}\right)^2$$

After subsitution we get:

$$C_p = -\frac{Q}{\pi u_\infty} \frac{1}{\left(x^2 + y^2\right)} \left(x + \frac{Q}{4\pi u_\infty} \right)$$

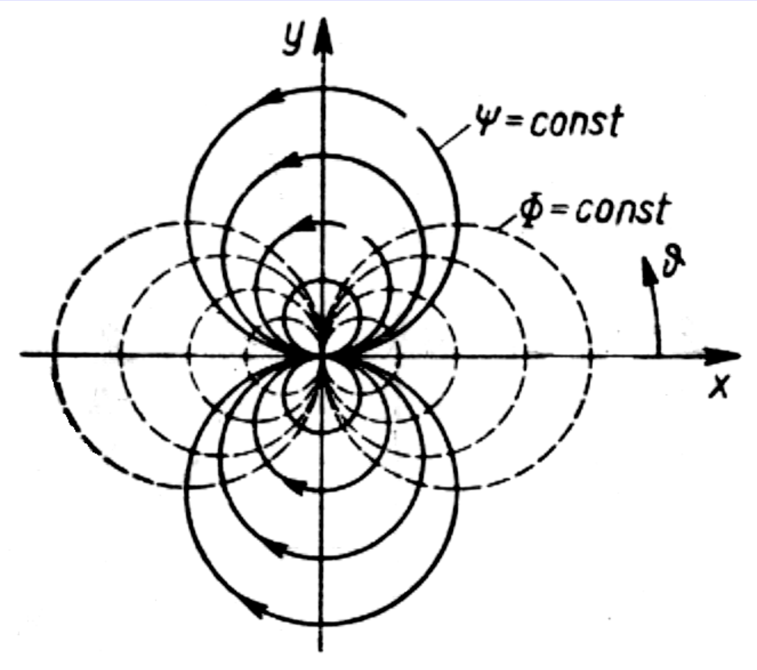
In the stagnation point we get:

$$C_p = -\frac{Q}{\pi u_\infty} \left(-\frac{2\pi u_\infty}{Q} + \frac{\pi u_\infty}{Q} \right) = 1$$

But on the zero-th stream line at the point $x=0$ we have:

$$u_x = u_\infty \qquad u_y = \pm \frac{2u_\infty}{\pi} \qquad \text{or:} \qquad u = \sqrt{u_\infty^2 + \frac{4u_\infty^2}{\pi^2}} \approx 1{,}184 u_\infty$$

3. The doublet (the dipole)



The dipole results from superposition of a positive and negative source of the same absolute intensity. The measure of the dipole intensity is the so called moment $M=2aQ$. Contrary to a source, a dipole has directional characteristics, because it emits fluid in a given direction and sucks in fluid from the opposite direction. Hence its location in space is important.

For a dipole at $x=0,\ y=0,$ directed in the positive x direction, we have:

The dipole potential:
$$\varphi = -\frac{M}{2\pi}\frac{x}{x^2+y^2}$$

The dipole stream function:
$$\psi = \frac{M}{2\pi}\frac{y}{x^2+y^2}$$

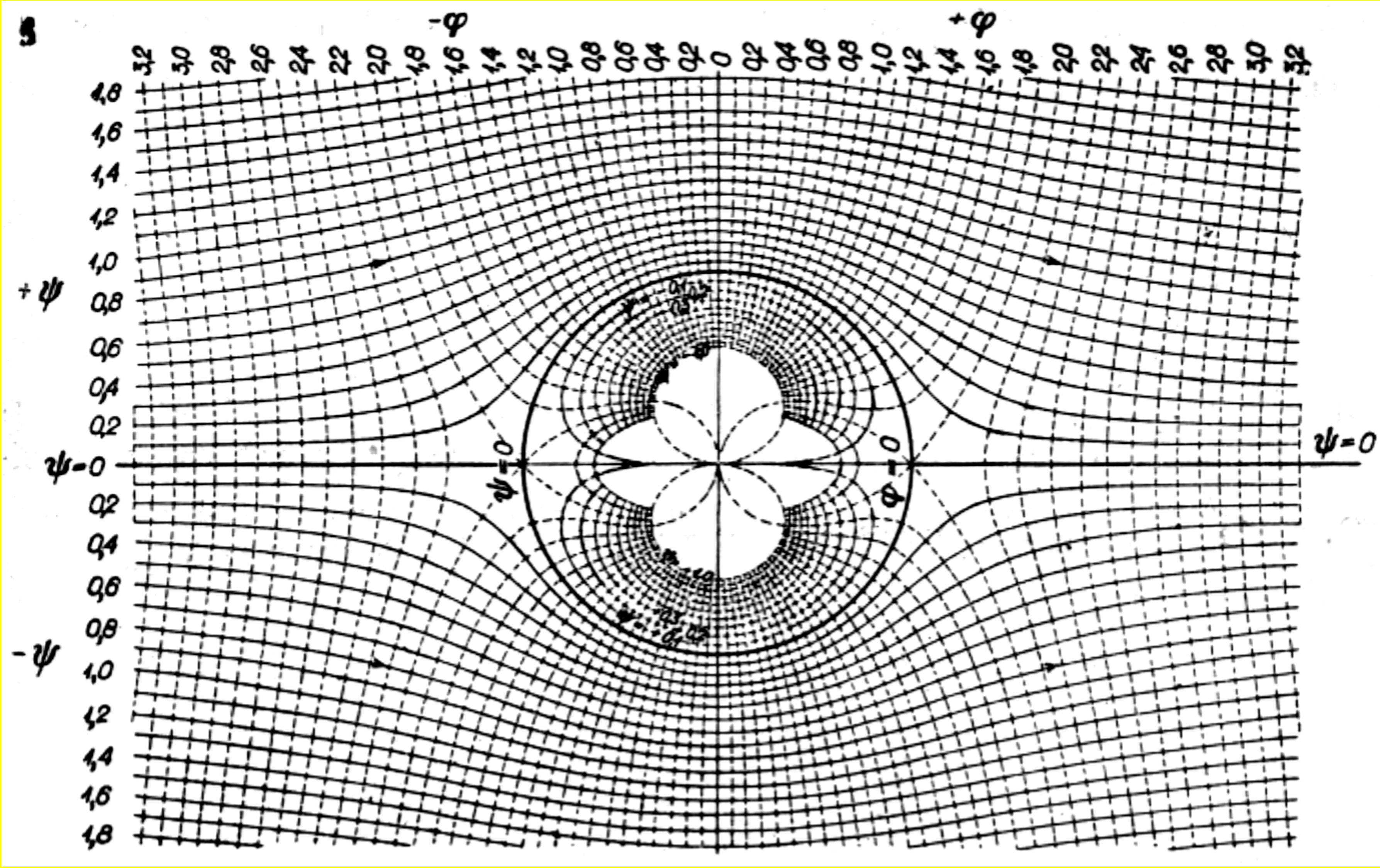
The stream lines for a dipole: $$\frac{y}{x^2+y^2}=C \rightarrow x^2+\left(y-\frac{C}{2}\right)=\frac{C^2}{4}$$

The stream lines are circles of radii *C/2*, the centres of which lie on the *y* axis in the points *y=C/2*.

The equipotential lines for a dipole: $$\frac{x}{x^2+y^2}=C \rightarrow \left(x-\frac{C}{2}\right)^2+y^2=\frac{C^2}{4}$$

The equipotential lines are circles of radii *C/2*, the centres of which lie on the *x* axis in the points *x=C/2*.

Example: Irrotational flow around a circular cylinder



Superposition: the uniform flow + the dipole

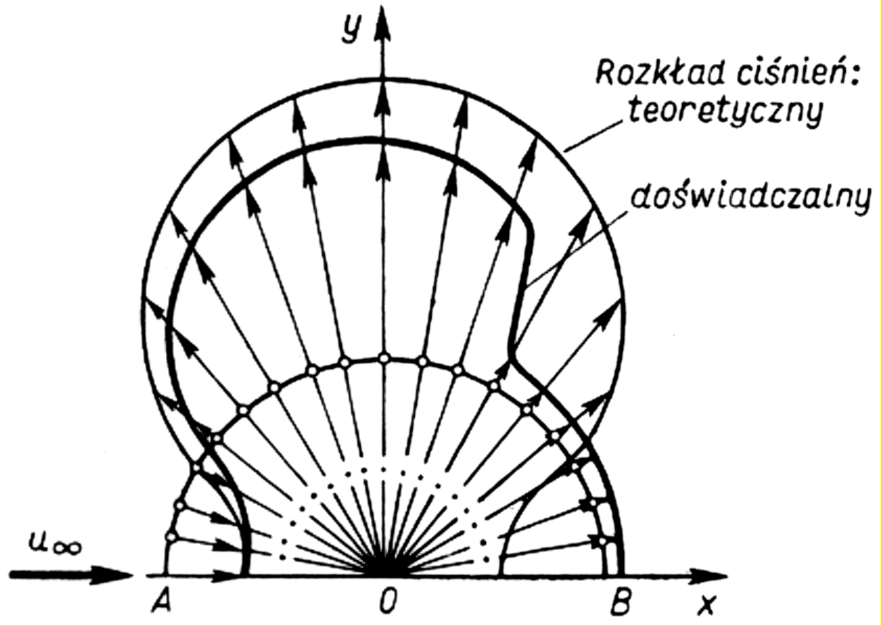
In the system $Or\theta$ we have: $M = 2\pi a^2 u_\infty$ where: a – cylinder radius

$$u = u_\infty \sqrt{1 + \left(\frac{a}{r}\right)^4 - 2\left(\frac{a}{r}\right)^2 \cos 2\theta}$$ where: $$\theta = arctg \frac{y}{x}$$

$$p = p_\infty + \frac{\rho u_\infty^2}{2}\left(1 - \frac{u^2}{u_\infty^2}\right)$$ for $r=a$ i.e. on the cylinder surface we have:

$$p_\theta = p_\infty + \frac{\rho u_\infty^2}{2}\left(1 - 4\sin^2\theta\right)$$

The components of the force acting on the cylinder may be calculated:



$$P_x = -a\int_0^{2\pi} p_\theta \cos\theta d\theta = 0$$

$$P_y = -a\int_0^{2\pi} p_\theta \sin\theta d\theta = 0$$

This is so called d'Alembert paradox

The d'Alembert paradox means, that in a potential flow the forces exerted by the flowing fluid on the immersed bodies are equal zero, what does not agree with a common experience. This is a direct consequence of the symmetry of the calculated pressure field, which is symmetrical both with respect to the *x* axis and *y* axis. In reality, the pressure field on the cylinder is asymmetrical with respect to the y axis, what is shown in the diagram on the previous slide and in the sketch and photograph below, showing the calculated and experimentally visualised flows. The differences are first of all visible on the trailing side of the cylinder.