J. Szantyr – Lecture No. 28 – Flows in open channels II

The flows in open channels often exhibit unsteady phenomena, first of all connected with the wave systems generated on the free liquid surface.

The simplest case concerns the propagation of so called small disturbances on the free surface. In such a case the viscosity forces may be neglected and equations of liquid motion may be linearized.

We can use the mass conservation equation:

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = 0$$

and the momentum conservation equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0$$

In the case of small disturbances of the layer thickness and velocity:

$$h = h_0 + h'(t,x) \qquad u = u_0 + u'(t,x)$$

$$h' << h_0 \qquad u' << u_0$$

they may be substituted into the equations, which are linearized to the form :

$$\frac{\partial h'}{\partial t} + u_0 \frac{\partial h'}{\partial x} + h_0 \frac{\partial u'}{\partial x} = 0 \qquad \frac{\partial u'}{\partial t} + u_0 \frac{\partial u'}{\partial x} + g \frac{\partial h'}{\partial x} = 0$$

The solution of this system of equations leads to:

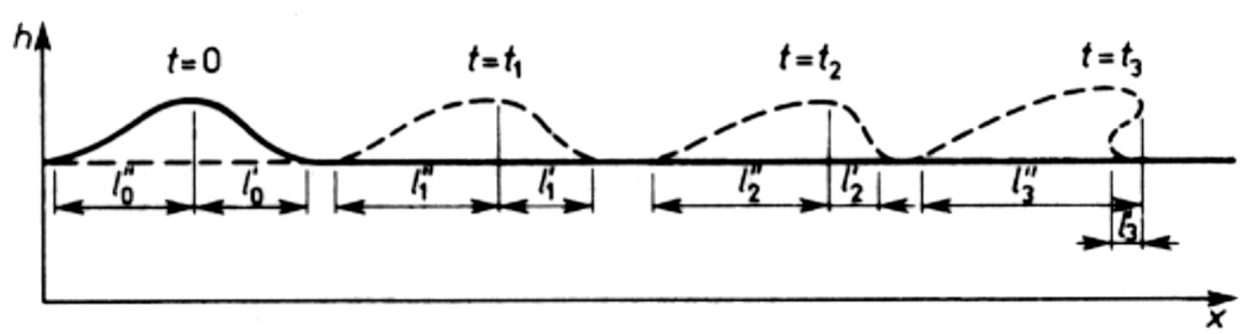
$$h' = h'\left[x - \left(u_0 \pm \sqrt{g h_0}\right) \cdot t\right] \qquad u' = u'\left[x - \left(u_0 \pm \sqrt{g h_0}\right) \cdot t\right]$$

The format of the solution shows that the small disturbances propagate with the characteristic velocity $\sqrt{g h_0}$ in the positive and negative x directions without changing their initial shape determined in the initial conditions $h'(0, x)$ and $u'(0, x)$

In the case of disturbances of the finite (large) amplitude the linearization of the equations of the liquid motion is no longer possible. Solution of the system of non-linear equations leads to much more complicated relations describing the propagation of such disturbances.

$$h = h\left[x - 2\sqrt{gh_0}\left(\frac{3}{2}\sqrt{\frac{h}{h_0}} - 1\right)\cdot t\right]$$

The above formula describes the propagation of a disturbance *h* in the positive direction of the x axis. It should be noticed that in this case the velocity is proportional to the square root of the disturbance amplitude. This leads to the self-deformation of the disturbance, as shown in the picture.

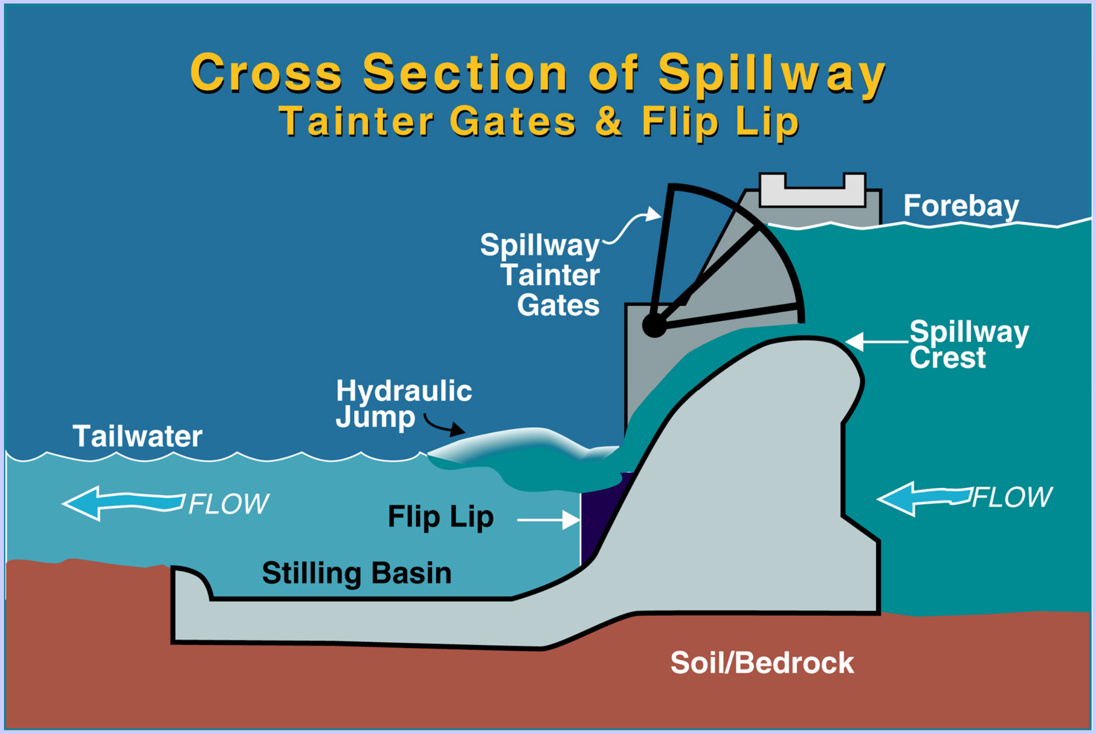


The wave front becomes more steep and its back – more dispersed. As the results the wave breaking occurs, which in a channel has the form of so called hydraulic jump. It should be remembered that in this simple, one-dimensional model the effects of dissipation, related to the internal liquid friction and the effects of dispersion, related to the liquid inertia in wave motion are both neglected.

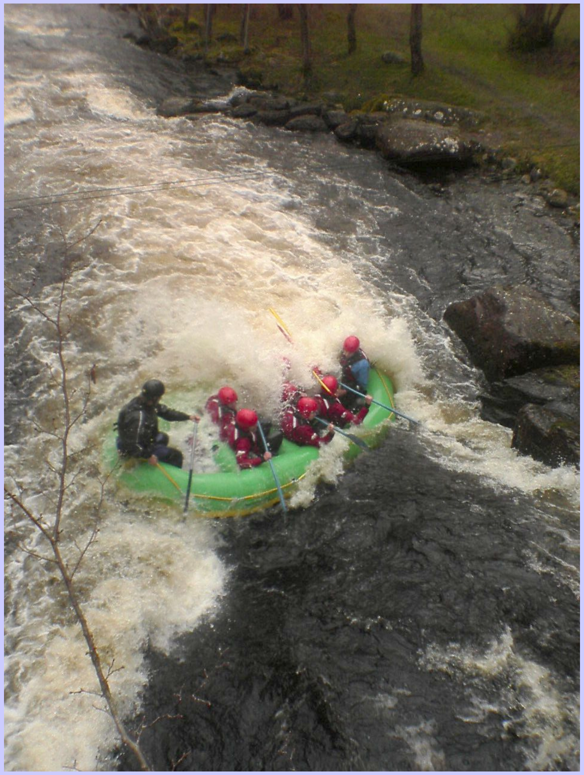
Hydraulic jump

The hydraulic jump is a complex unsteady phenomenon, taking place in supercritical flows.

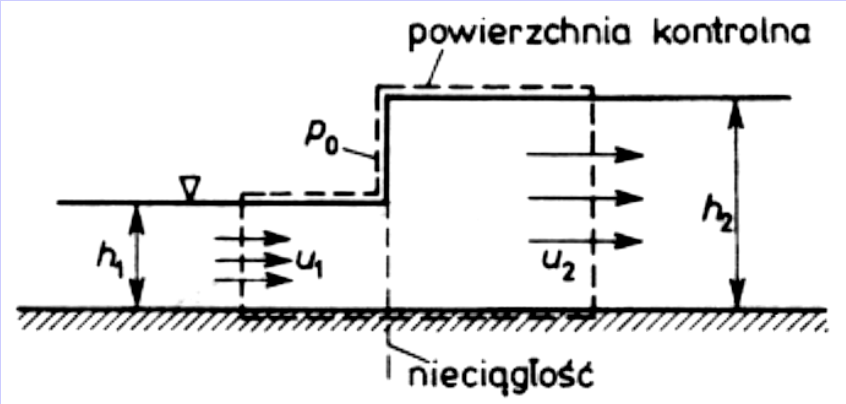
The hydraulic jump causes an abrupt increase in the liquid layer thickness. It may be easily generated in any wash basin.



Examples of the hydraulic jumps.



A simple theoretical analysis of the hydraulic jump



Mass conservation equation:

$$\rho h_1 u_1 = \rho h_2 u_2 = m$$

Momentum conservation equation:

$$p_1 = p_0 + \rho g \left(h_1 - z \right)$$
$$p_2 = p_0 + \rho g \left(h_2 - z \right)$$

in the vertical direction

$$m\left(u_2 - u_1\right) = \int_0^{h_1} p_1 dz - \int_0^{h_2} p_2 dz + \int_{h_1}^{h_2} p_0 dz$$

in the horizontal direction

After substitution of pressures and integration we get:

$$h_1 u_1^2 + \frac{1}{2} g h_1^2 = h_2 u_2^2 + \frac{1}{2} g h_2^2$$

If we substitute: $u_2 = u_1 \dfrac{h_1}{h_2}$ $\qquad u_1 = Fr_1 \cdot \sqrt{g h_1}$

then we get: $$\left[\left(\frac{h_2}{h_1}\right)^2 + \frac{h_2}{h_1} - 2\cdot Fr_1^2\right]\left(\frac{h_2}{h_1} - 1\right) = 0$$

This equation has three solutions, but only one of them is physically realistic:

$$\boxed{\frac{h_2}{h_1} = \frac{\sqrt{1+8\cdot Fr_1^2} - 1}{2}}$$

This solution means that:

for: $Fr_1 = 1$ there is $h_2 = h_1$

for: $Fr_1 > 1$ there is $h_2 > h_1$ and $Fr_2 < 1$

for: $Fr_1 < 1$ there is $h_2 < h_1$

It follows from the above that the hydraulic jump is possible only in the supercritical flow. The analysis of the energy conservation equation leads to the conclusion that in the hydraulic jump a marked increase in the liquid internal energy, due to intensive dissipation processes, takes place.

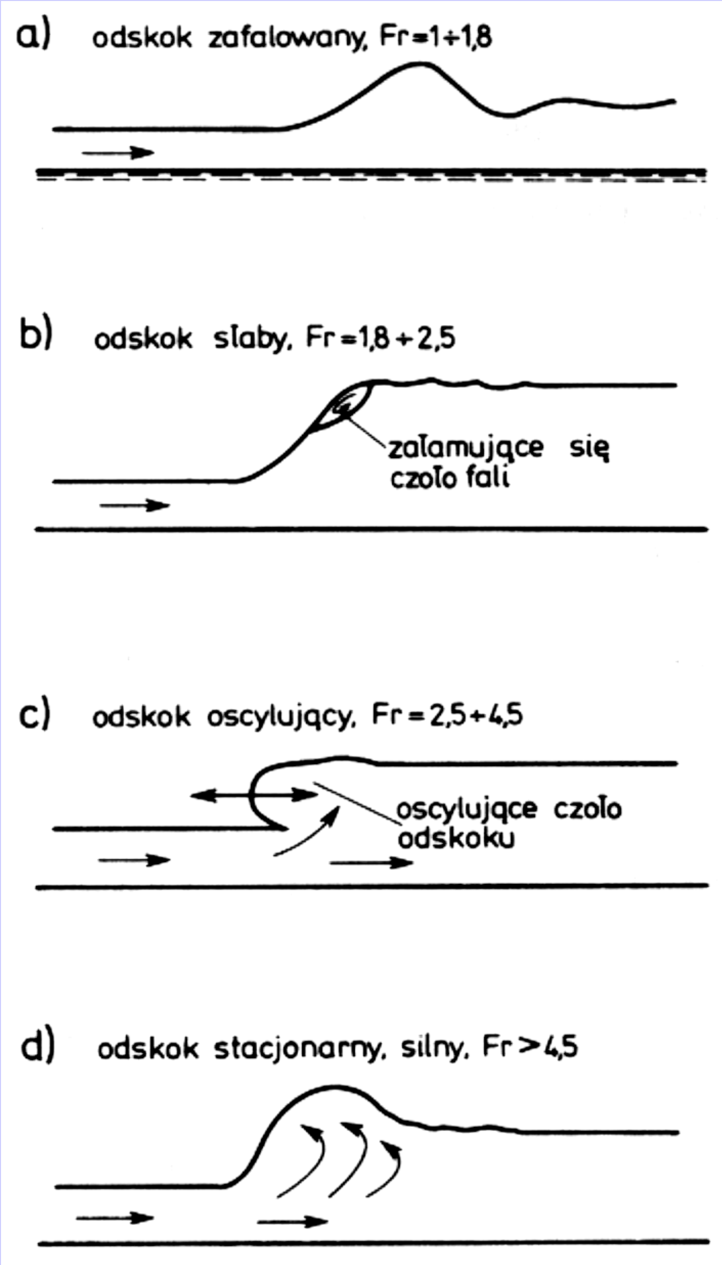
The loss of the mechanical energy of the flowing liquid due to the dissipation processes in the hydraulic jump may be calculated according to the following formula:

$$\frac{\Delta E}{E_1} = \frac{\left(\sqrt{1+8\cdot Fr_1^2}-3\right)^3}{8\left(\sqrt{1+8\cdot Fr_1^2}-1\right)\left(\sqrt{Fr_1^2+2}\right)}$$

The hydraulic jumps may be generated intentionally, because of the following reasons:

- dissipation of the liquid mechanical energy
- increase in the liquid level
- mixing of the additives in the liquid
- aeration of the liquid

The form of the hydraulic jump depends on the Froude number



a standing or undulating jump

$$\frac{h_2}{h_1} = 1 \leftrightarrow 2 \qquad \frac{\Delta E}{E_1} < 0.05$$

a weak jump

$$\frac{h_2}{h_1} = 2 \leftrightarrow 3.1 \qquad \frac{\Delta E}{E_1} = 0.05 \leftrightarrow 0.15$$

an oscillating jump

$$\frac{h_2}{h_1} = 3.1 \leftrightarrow 5.9 \qquad \frac{\Delta E}{E_1} = 0.15 \leftrightarrow 0.45$$

a stable jump

$$\frac{h_2}{h_1} = 5.9 \leftrightarrow 12 \qquad \frac{\Delta E}{E_1} = 0.45 \leftrightarrow 0.70$$

Example

In the channel of depth h=1.0 [m] the water flow of mean velocity u=5.0 [m/s] has been generated. Will the hydraulic jump occur in this flow? If so, what will be the water depth behind the jump and what will be the mean flow velocity there?

Solution

$$Fr_1 = \frac{u}{\sqrt{gh}} = \frac{5.0}{\sqrt{9.81 \cdot 1.0}} \approx 1.6 > 1$$ The wavy jump will occur

$$h_2 = h_1 \frac{\sqrt{1 + 8 \cdot Fr_1^2} - 1}{2} = 1.0 \frac{\sqrt{1.0 + 8 \cdot 1.6} - 1.0}{2} \approx 1.36[m]$$

$$u_2 = u_1 \frac{h_1}{h_2} = 5.0 \frac{1.0}{1.36} \approx 3.68[m/s]$$