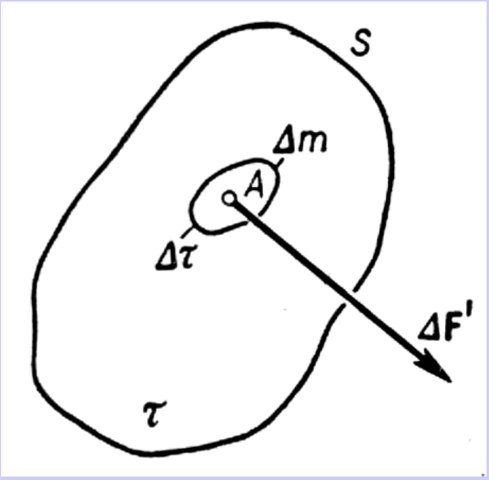
J. Szantyr – Lecture No. 3 – Fluid in equilibrium

Internal forces – mutual interactions of the selected mass elements of the analysed region of fluid, forces having a surface character, forming pairs acting in the opposite directions, thus reducing to zero.

External forces – the result of action of masses which do not belong to the analysed region of fluid – they may be divided into mass forces and surface forces.

Mass forces act on every fluid element and they are proportional to its mass.

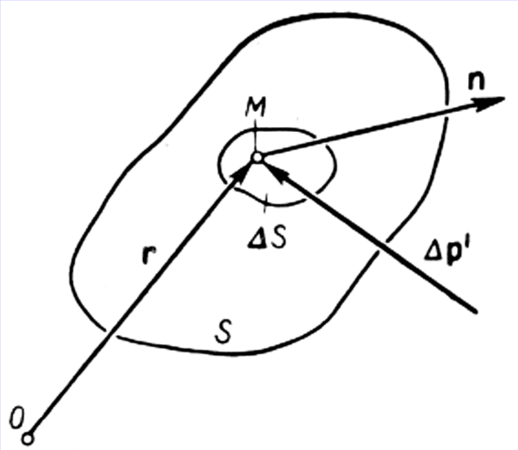


$$\overline{F} = \lim_{\Delta m \to 0} \frac{\Delta \overline{F'}}{\Delta m} = \frac{1}{\rho} \lim_{\Delta \tau \to 0} \frac{\Delta \overline{F'}}{\Delta \tau} = \frac{1}{\rho} \frac{dF'}{d\tau}$$

$\overline{F}\left[\frac{m}{s^2}\right]$ unit mass force, eg. gravitational force, in other words acceleration g

$\rho\left[\frac{kg}{m^3}\right]$ density of fluid

Surface forces act on the surface surrounding the selected region of fluid and they are proportional to the area of this surface.



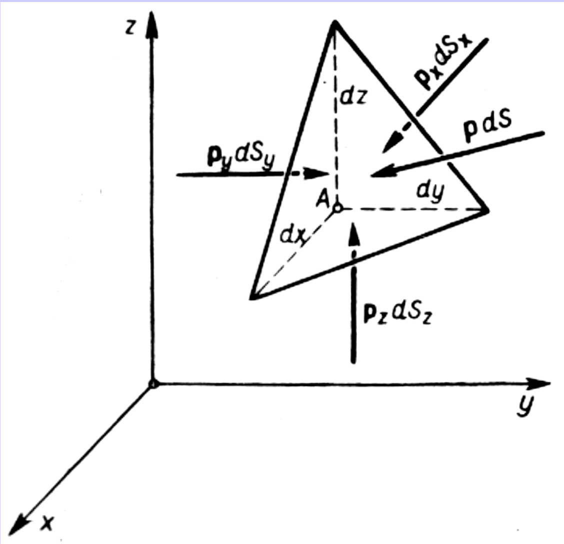
$$\overline{P} = \lim_{\Delta s \to 0} \frac{\Delta \overline{p'}}{\Delta s} = \frac{d\overline{p'}}{ds}$$

$\overline{P}\left[\frac{N}{m^2}\right]$ unit surface force

In general the surface force depends on the orientation of the surface element, defined by the unit length normal vector *n,* thus it should be symbolized by: P_n

The fluid remains in equilibrium under the action of the given external forces if the forces acting on an arbitrarily selected part of the fluid form the system of vectors equivalent to zero.

In the fluid in the state of equilibrium the pressure in an arbitrary point has a constant value and it does not depend on the orientation of the surface element passing through this point.



Equilibrium conditions of the tetrahedron:

$$p_x dS_x - p dS \cos(\overline{p}, x) = 0$$

$$p_y dS_y - p dS \cos(\overline{p}, y) = 0$$

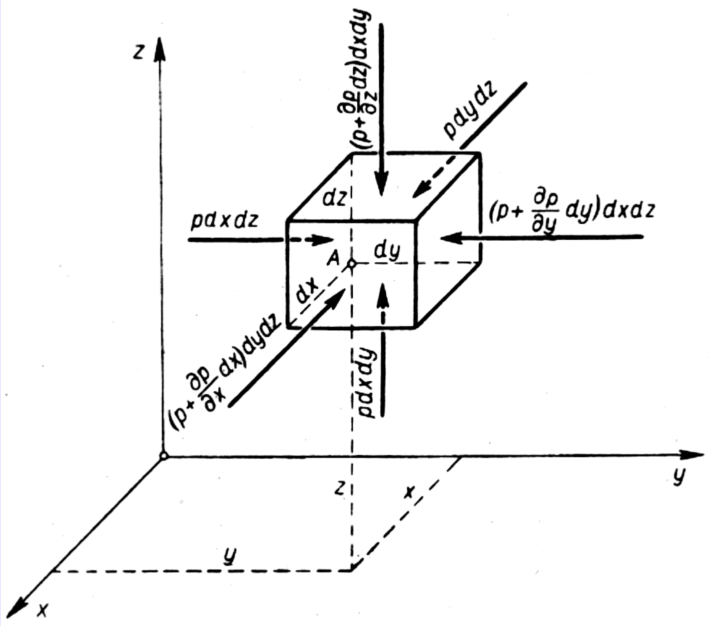
$$p_z dS_z - p dS \cos(\overline{p}, z) = 0$$

but we have: $dS_x = dS \cos(\overline{p}, x)$ etc.

$p_x - p = 0; p_y - p = 0; p_z - p = 0$ hence: $p = p_x = p_y = p_z$

Conclusion: the hydrostatic state of stress in the fluid has the character of a scalar field.

Conditions of the fluid equilibrium



Unit mass force:

$$\bar{F} = X\bar{i} + Y\bar{j} + Z\bar{k} = \bar{F}(x, y, z)$$

Density:

$$\rho = \rho(x, y, z)$$

Conditions of equilibrium of the fluid element:

$$X\rho dxdydz + pdydz - \left(p + \frac{\partial p}{\partial x}dx \right)dydz = 0$$

$$Y\rho dxdydz + pdxdz - \left(p + \frac{\partial p}{\partial y}dy \right)dxdz = 0$$

$$Z\rho dxdydz + pdxdy - \left(p + \frac{\partial p}{\partial z}dz \right)dxdy = 0$$

hence we obtain: $X = \frac{1}{\rho}\frac{\partial p}{\partial x}$ $Y = \frac{1}{\rho}\frac{\partial p}{\partial y}$ $Z = \frac{1}{\rho}\frac{\partial p}{\partial z}$

what leads to the basic hydrostatic **Euler equation:**

$$\overline{F} = \frac{1}{\rho} gradp$$

or in the differential form:

$$Xdx + Ydy + Zdz = \frac{dp}{\rho}$$

if the mass force field has the potential U, such that: $\overline{F} = -gradU$

then we obtain: $dU = -\frac{dp}{\rho}$ and after integration: $p = -\rho U + C$

The integration constant may be determined from the given pressure and mass force field potential in the selected point in the fluid.

For example in the gravitational field near the Earth we have $X=Y=0$

$Z = -g = -\frac{\partial U}{\partial z}$ or: $U = gz$ what gives: $p = -\rho g z + C$

Conclusion: in the gravitational field of the Earth the surfaces of constant hydrostatic pressure (isobaric surfaces) are horizontal.

General conclusion: isobaric and equipotential surfaces are perpendicular to the vector of mass forces (see example at the end)

If by p_a we denote the pressure on the fluid free surface at the elevation H we get:

$p_a = -\rho g H + C$ what gives: $C = p_a + \rho g H$ and further:

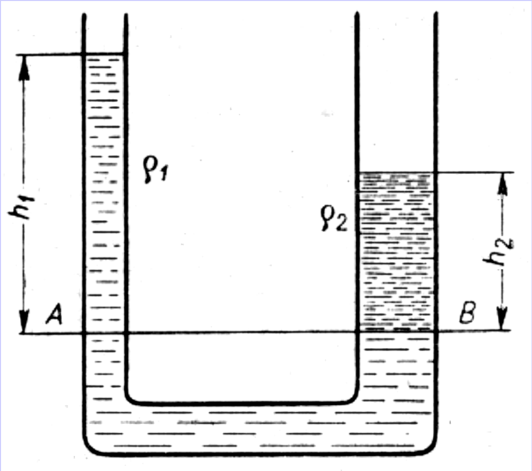
$p = p_a + \rho g (H - z)$ finally: $p = p_a + \rho g h$ where:

$h=H-z$ – immersion of the point under free surface

p_a - pressure on the free surface (eg. atmospheric)

Examples of application

Connected vessels – at the level A-B we have:

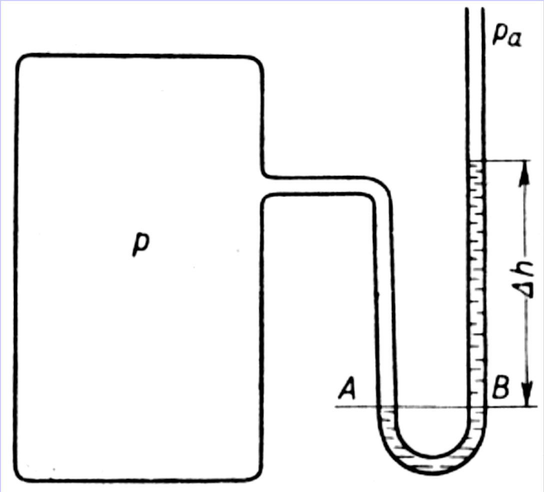


$$p = p_a + \rho_1 g h_1 \qquad p = p_a + \rho_2 g h_2$$

czyli:

$$\rho_1 h_1 = \rho_2 h_2$$ albo: $$\frac{h_1}{h_2} = \frac{\rho_2}{\rho_1}$$

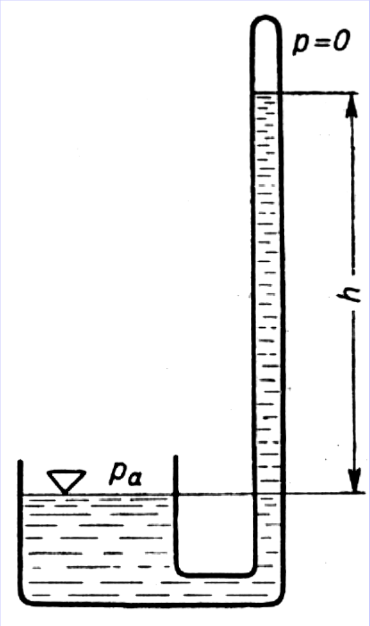
Hydrostatic measurement of pressure



$$p_A = p \qquad p_B = p_a + \rho g \Delta h$$

$$p_A = p_B \qquad p - p_a = \rho g \Delta h$$

In this way we measure the overpressure (relative pressure) in the container, or the difference between the absolute pressure *p* and the atmospheric pressure.

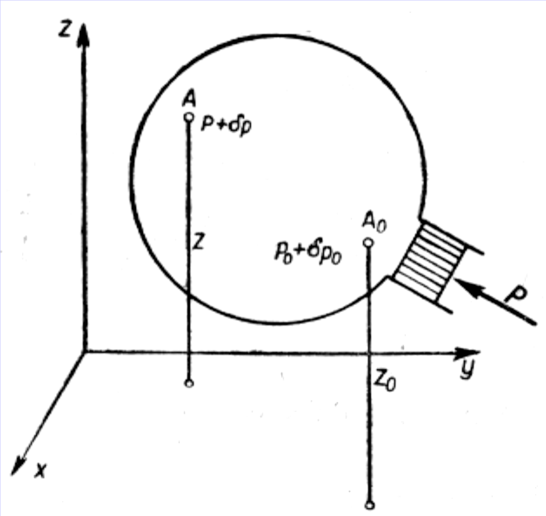


Barometer – measurement of the atmospheric pressure.

$$p_a = \rho g h$$

Pascal theorem

Increase of pressure in an arbitrary point of the homogenous incompressible fluid in the state of equilibrium, placed in the potential mass force field, results in the same increase of pressure in any other point of the fluid.



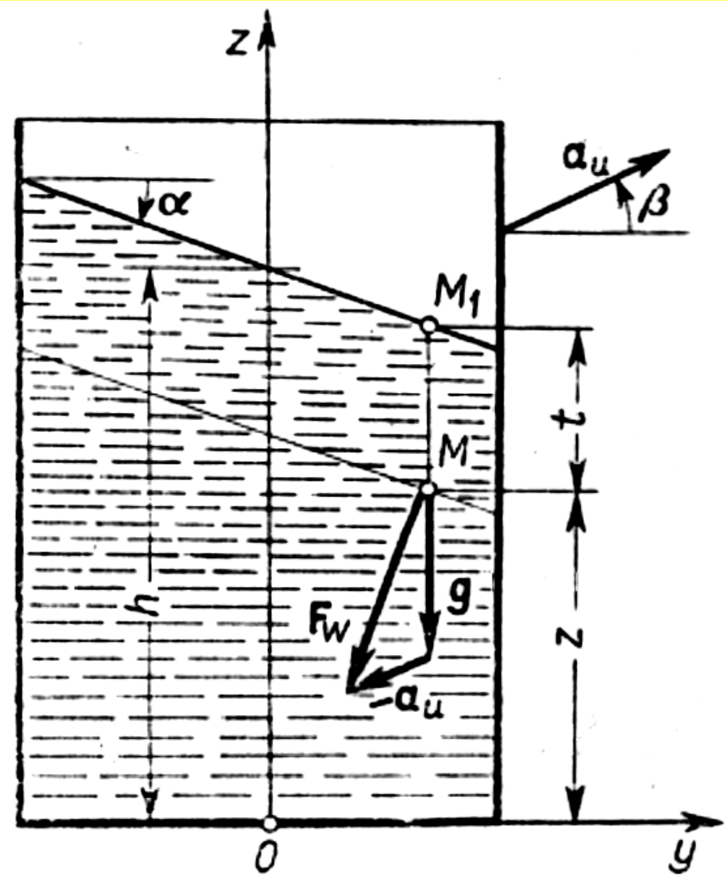
$$p - p_0 = \rho\left(U_0 - U\right)$$

$$p + \delta p - \left(p_0 + \delta p_0\right) = \rho\left(U_0 - U\right)$$

$$\delta p - \delta p_0 = 0$$

$$\delta p = \delta p_0$$

Example No.1: Determination of the inclination of the liquid free surface in the container moving in a straight constantly accelerated motion in an arbitrary direction.



$\bar{a}_u$ convective acceleration $a = |\bar{a}_u|$

projections of the unit mass force:

$$X = 0$$
$$Y = -a\cos\beta$$
$$Z = -g - a\sin\beta$$

liquid equilibrium equations:

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$
$$\frac{1}{\rho}\frac{\partial p}{\partial y} = -a\cos\beta$$
$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g - a\sin\beta$$

or: $$\frac{dp}{\rho} = -\left(a\cos\beta dy + a\sin\beta dz + gdz\right)$$

after integration with $\rho = const$ we obtain:

$$p = -\rho g\left[\frac{a}{g}y\cos\beta + \left(\frac{a}{g}\sin\beta + 1\right)z\right] + C_1$$

the constant is determined for the pressure at the free surface in the point M1:

$$C_1 = p_a + \rho g\left[\frac{a}{g}y\cos\beta + \left(\frac{a}{g}\sin\beta + 1\right)(z + t)\right]$$

after substituting we get:

$$p = p_a + \rho g\left(1 + \frac{a}{g}\sin\beta\right)t$$

the equation of the isobaric (constant pressure) surfaces has the form:

$$z = -\frac{\cos\beta}{\frac{a}{g} + \sin\beta} y + C$$

it describes a family of planes inclined at an angle α such that:

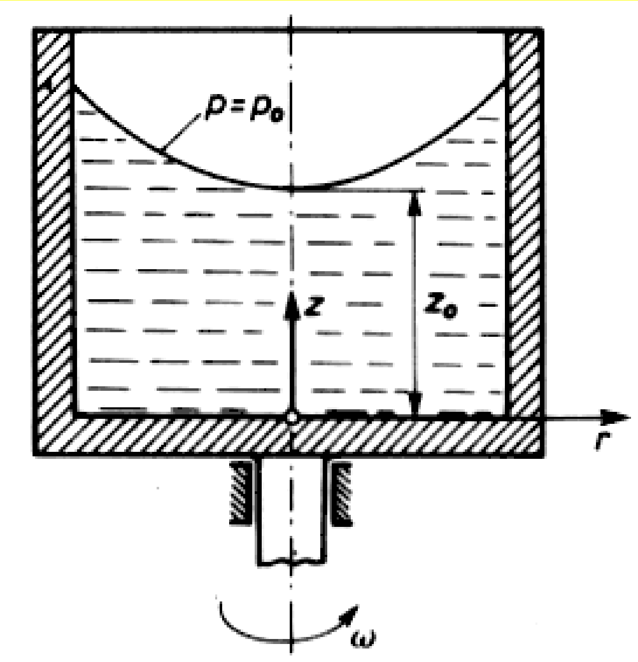
$$tg\alpha = -\frac{\cos\beta}{\frac{a}{g} + \sin\beta}$$

but the angle of inclination of the resultant mass force φ is:

$$tg\varphi = \frac{Z}{Y} = \frac{\frac{a}{g} + \sin\beta}{\cos\beta} = ctg\alpha$$

Conclusion: the resultant mass force is perpendicular to the isobaric surfaces.

Example No. 2: Determination of the relation describing the pressure distribution in a tank rotating with constant angular velocity ω. The tank is filled with liquid having density ρ, and the ambient pressure is equal p.



In the cylindrical system of co-ordinates the basic hydrostatic equation has the form:

$$\frac{dp}{\rho} = q_r \cdot dr + q_\vartheta \cdot r \cdot d\vartheta + q_z \cdot dz$$

where the respective terms are equal to:

$$q_r = \omega^2 \cdot r$$

$$q_z = -g$$

$$q_\vartheta = 0$$

After substitution we get: $dp = \rho \cdot \left(\omega^2 \cdot r \cdot dr - g \cdot dz\right)$

Integration leads to: $p = \frac{\rho}{2} \cdot \omega^2 \cdot r^2 - \rho \cdot g \cdot z + C$

For the liquid surface point at the tank axis we have:

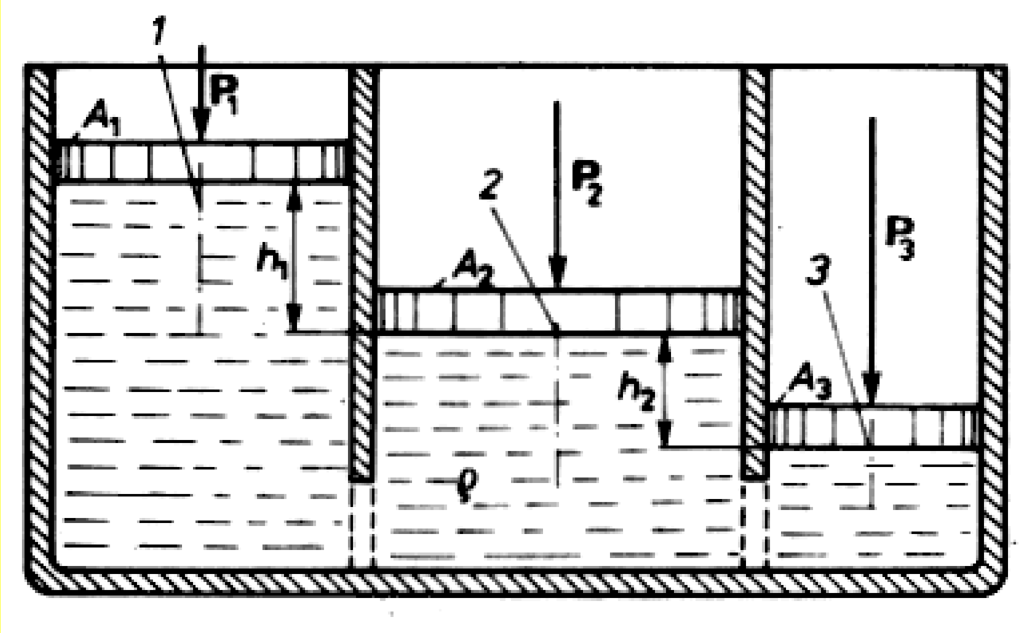
$r = 0$ $z = z_0$ $p = p_0$

Hence the integration constant is equal to: $C = p_0 + \rho \cdot g \cdot z_0$

Finally, the pressure distribution in the liquid is described by equation:

$$p = p_0 - \rho \cdot g \cdot \left(z - z_0\right) + \frac{\rho}{2} \cdot \omega^2 \cdot r^2$$

Example No. 3: Three pistons of areas A1=0,6 m2, A2=0,8 m**2, A3 0,4 m**2, respectively loaded with forces P1=1 kN, P2=2 kN i P3=3 kN, act on water having density ρ=1000 kg/m**3. Determine at which elevations h1 i h2 the system of pistons remains in equilibrium.**



Pressure under piston 2 is:

$$\frac{P_1}{A_1} + h_1 \cdot \rho \cdot g = \frac{P_2}{A_2}$$

Pressure under piston 3 is:

$$\frac{P_2}{A_2} + h_2 \cdot \rho \cdot g = \frac{P_3}{A_3}$$

The elevations are determined from the above equations:

$$h_1 = \left(\frac{P_2}{A_2} - \frac{P_1}{A_1} \right) \cdot \frac{1}{\rho \cdot g} = 0.085[m] \qquad h_2 = \left(\frac{P_3}{A_3} - \frac{P_2}{A_2} \right) \cdot \frac{1}{\rho \cdot g} = 0.51[m]$$

Correctness of the solution may be checked using the equilibrium equation referring to pistons 1 and 3:

$$\frac{P_1}{A_1} + \rho \cdot g \cdot (h_1 + h_2) = \frac{P_3}{A_3} \rightarrow \frac{1000}{0.6} + 1000 \cdot 9.81 \cdot (0.085 + 0.51) = \frac{3000}{0.4} \rightarrow 7500 = 7500$$