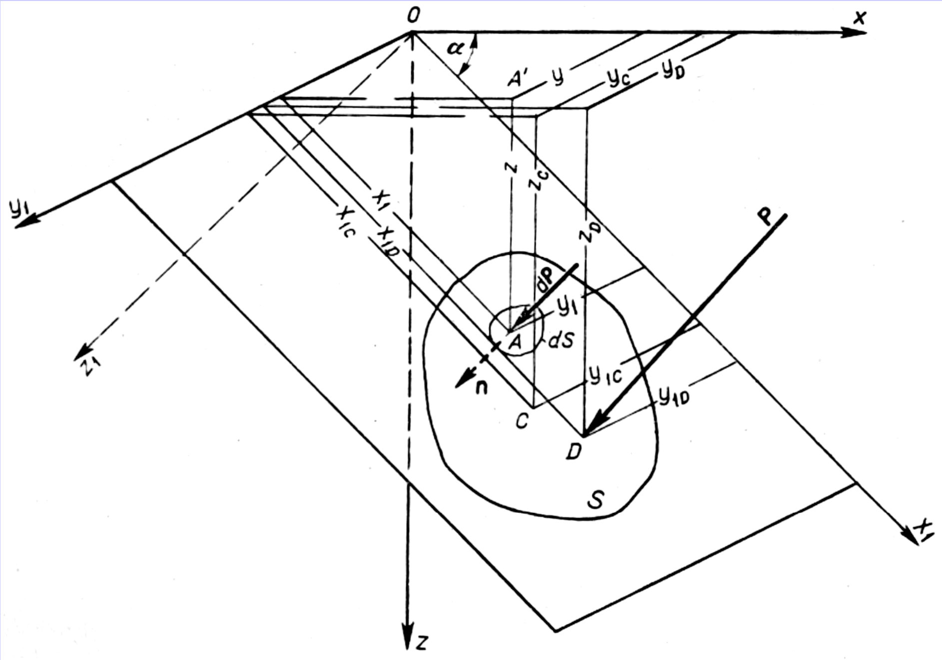
J. Szantyr – Lecture No. 4 – Hydrostatic force

Hydrostatic force acting on a flat wall



Hydrostatic force acting on a flat wall has the form of a system of parallel elementary forces perpendicular to the wall. It may be reduced to the resultant force equal to their sum and acting at the centre of action of the parallel forces.

Elementary force: $$d\overline{P} = \overline{n}\left(p - p_a\right)dS = \overline{n}\rho g z dS$$

Total force: $$\boxed{\overline{P} = \rho g \int_S \overline{n} z dS = \rho g \overline{n} \int_S z dS = \rho g \overline{n} z_C S}$$

where: z_C - immersion of the geometric centre of the wall S

The hydrostatic force on the flat wall of an arbitrary outline and arbitrarily inclined to the horizon is equal (in its module) to the weight of the prism of fluid having the base S and height equal to the immersion of the geometrical centre of S under the free surface.

Projections of the hydrostatic force on the axes of the system Oxyz:

$$P_x = \rho g \int_S z \cos(\bar{n}, \bar{i}) dS = -\rho g \sin\alpha \int_S z dS = -\rho g z_C S \sin\alpha$$

$$P_y = \rho g \int_S z \cos(\bar{n}, \bar{j}) dS = 0$$

$$P_Z = \rho g \int_S z \cos(\bar{n}, \bar{k}) dS = \rho g \cos\alpha \int_S z dS = \rho g z_C S \cos\alpha$$

Hence the module of the force: $P = \sqrt{P_x^2 + P_z^2} = \rho g z_C S$

Determination of the point of action of the resultant hydrostatic force

Moment of the force:

$$\overline{M}_D = \overline{r}_D \times \overline{P} = \rho g \int_S \overline{r}_D \times \overline{n} z dS = \overline{i}_1 \rho g \int_S y_1 z dS + \overline{j}_1 \rho g \int_S x_1 z dS$$

Projections of the main vector of the moment:

$$x_{1D} P = \rho g \int_S x_1 z dS \qquad y_{1D} P = \rho g \int_S y_1 z dS$$

Co-ordinates of the point of action:

$$x_{1D} = \frac{\int_S x_1 z dS}{\int_S z dS} = \frac{\int_S x_1^2 dS}{\int_S x_1 dS} = \frac{\int_S x_1^2 dS}{x_{1C} S} = \frac{I_{y_1}}{x_{1C} S} \qquad y_{1C} = \frac{\int_S y_1 z dS}{\int_S z dS} = \frac{\int_S x_1 y_1 dS}{\int_S x_1 dS} = \frac{\int_S x_1 y_1 dS}{x_{1C} S} = \frac{D_{z_1}}{x_{1C} S}$$

where it was substituted: $z = x_1 \cos\alpha$

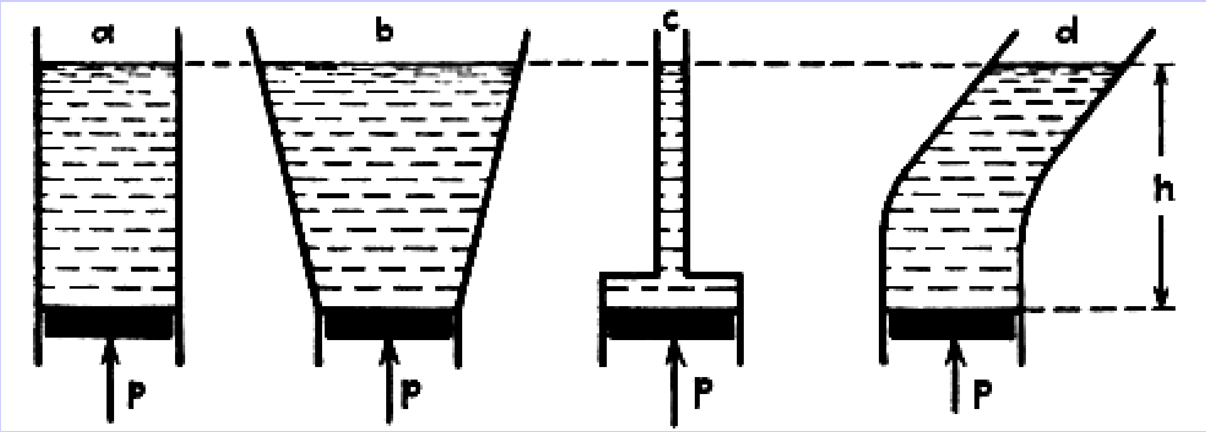
Determination of the point of action of the hydrostatic force requires computation of the moment of inertia of the wall S and computation of its centre of gravity.

Conclusions

Location of the point of action of the hydrostatic force in the system linked to the wall does not depend on the wall inclination.

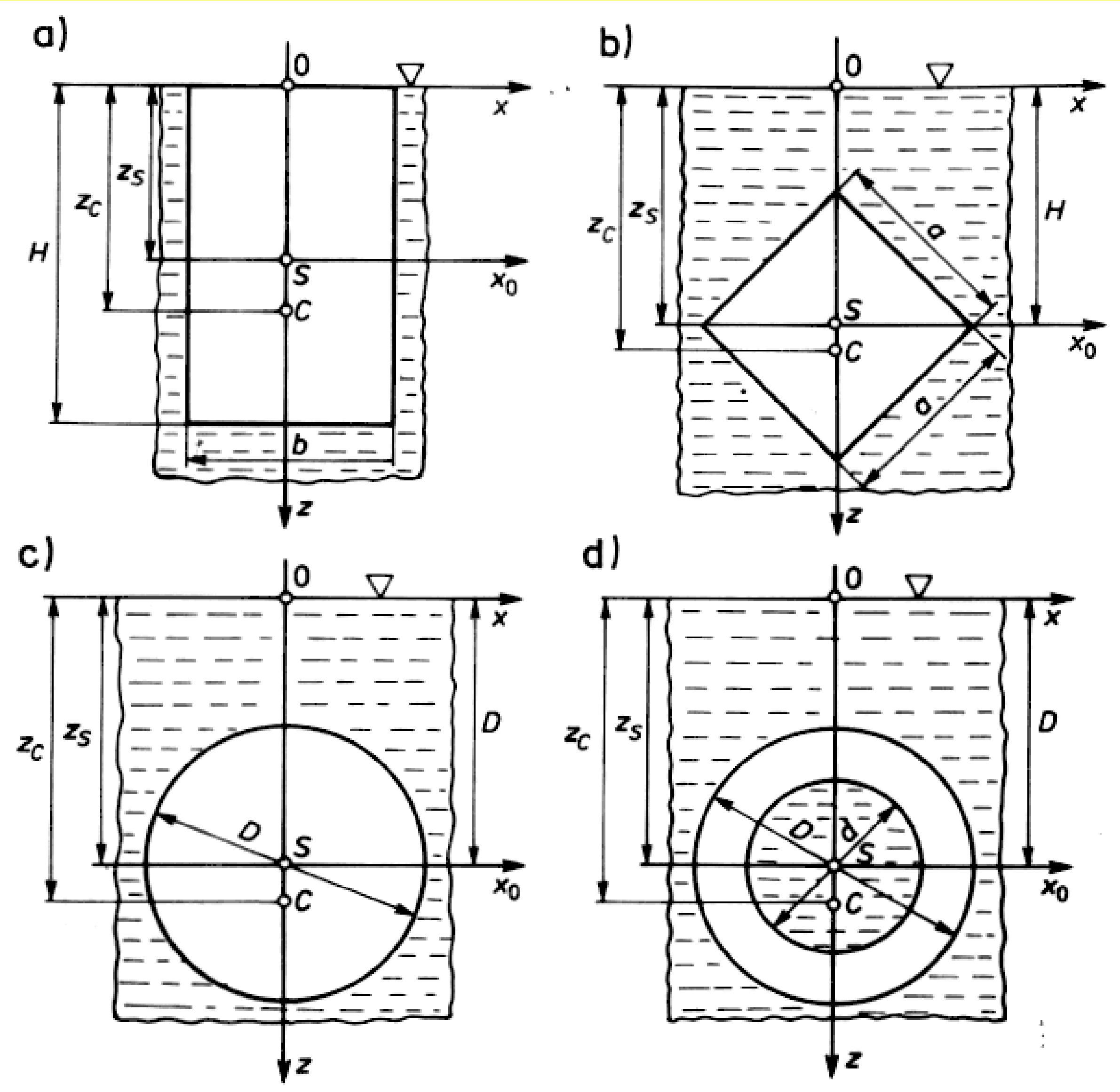
On horizontal and inclined walls the centre of action of the hydrostatic force lies below the geometric centre of the wall.

The magnitude of the hydrostatic force does not depend on the shape of the container.



In all above containers the force on the bottom is the same.

Example No. 1: determine the hydrostatic force and location of its point of action C for vertical walls shown in the picture below.



Solution for the wall „a”

Immersion of the geometric wall centre:

$$z_S = \frac{H}{2}$$

Wall area:

$$S = bH$$

Wall moment of inertia with respect to the main central axis:

$$I_{x0} = \frac{bH^3}{12}$$

Wall moment of inertia with respect to x axis (Steiner’s rule):

$$I_x = I_{x0} + z_S^2 S = \frac{bH^3}{12} + \frac{H^2}{4} bH$$

Point of action elevation:

$$z_C = \frac{I_x}{z_S S} = \frac{H}{2} + \frac{H}{6} = \frac{2}{3} H$$

Hydrostatic force module:

$$P = \rho g z_S S = \rho g \frac{H}{2} bH = \frac{\rho g b H^2}{2}$$

Solution for the wall „b”

Immersion of the geometric wall centre: $z_S = H$

Wall area: $S = a^2$

Wall moment of inertia with respect to the main central axis: $I_{x0} = \dfrac{a^4}{12}$

Wall moment of inertia with respect to x axis (Steiner’s rule): $I_x = I_{x0} + z_S^2 S = \dfrac{a^4}{12} + H^2 a^2$

Point of action elevation: $z_C = \dfrac{I_x}{z_S S} = H + \dfrac{a^4}{12a^2 H} = H + \dfrac{a^2}{12H}$

Hydrostatic force module: $P = \rho g z_S S = \rho g H a^2$

Solution for the wall „c”

Immersion of the geometric wall centre: $z_S = D$

Wall area: $$S = \frac{\pi D^2}{4}$$

Wall moment of inertia with respect to the main central axis: $$I_{x0} = \frac{\pi D^4}{64}$$

Wall moment of inertia with respect to x axis(Steiner’s rule): $$I_x = I_{x0} + z_S^2 S = \frac{\pi D^4}{64} + D^2 \frac{\pi D^2}{4}$$

Point of action elevation: $$z_C = \frac{I_x}{z_S S} = D + \frac{D}{16} = \frac{17}{16} D$$

Hydrostatic force module: $$P = \rho g z_S S = \rho g D \frac{\pi D^2}{4} = \frac{\pi \rho g D^3}{4}$$

Solution for the wall „d”

Immersion of the geometric wall centre: $$z_S = D$$

Wall area: $$S = \frac{\pi\left(D^2 - d^2\right)}{4}$$

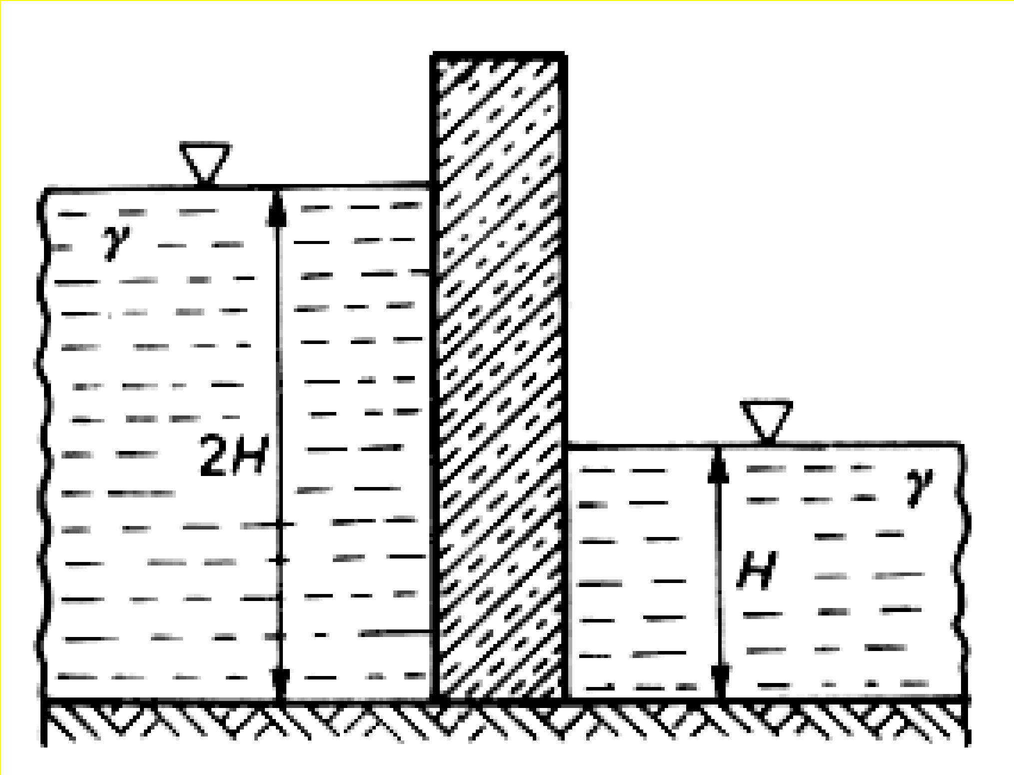
Wall moment of inertia with respect o the main central axis: $$I_{x0} = \frac{\pi\left(D^4 - d^4\right)}{64}$$

Wall moment of inertia with respect to x axis: $$I_x = I_{x0} + z_S^2 S = \frac{\pi\left(D^4 - d^4\right)}{64} + \frac{\pi\left(D^2 - d^2\right)}{4} D^2$$

Point of action elevation: $$z_C = \frac{I_x}{z_S S} = D + \frac{D^2 + d^2}{16D}$$

Hydrostatic force module: $$P = \rho g z_S S = \rho g \frac{\pi\left(D^2 - d^2\right)}{4} D$$

Example No. 2: Determine the hydrostatic moment acting of a vertical dam of width L with respect to its foot. The dam divides a canal of rectangular cross-section. On the left side the liquid level is 2H, on the right side the liquid level is H.



Solution

Hydrostatic forces on the left and right side are respectively equal to:

$$P_L = \rho g A_L z_{SL} \qquad\qquad P_P = \rho g A_P z_{SP}$$

Assuming the dam width L and knowing that:

$$z_{SL} = H \qquad\qquad z_{SP} = \frac{H}{2}$$

We get:

$$P_L = 2\rho g L H^2 \qquad\qquad P_P = \frac{1}{2}\rho g L H^2$$

Points of action of the hydrostatic forces may be determined on the basis of the preceding example for a rectangular wall:

$$z_{CL} = \frac{4}{3}H \qquad\qquad z_{CP} = \frac{2}{3}H$$

Moment acting on the dam is equal to: $M = P_L z_L - P_P z_P$

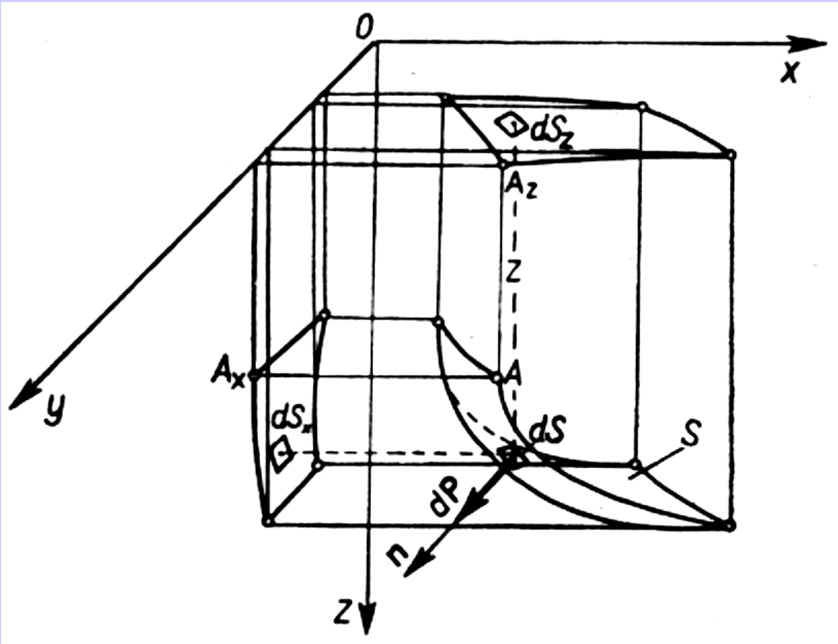
where: $$z_L = 2H - z_{CL} = 2H - \frac{4}{3}H = \frac{2}{3}H$$

$$z_P = H - z_{CP} = H - \frac{2}{3}H = \frac{1}{3}H$$

Finally, after substitution we get:

$$M = 2\rho g L H^2 \cdot \frac{2}{3}H - \frac{1}{2}\rho g L H^2 \cdot \frac{1}{3}H = \frac{7}{6}\rho g L H^3$$

Hydrostatic force on curved walls.



All elementary forces acting on the wall S form a spatial system of forces, which may be reduced to the main force vector and the main moment vector.

Elementary force:

$$d\overline{P} = \overline{n}\left(p - p_a\right)dS = \overline{n}\rho g z dS$$

Main force vector:

$$\overline{P} = \rho g \int_S \overline{n} z dS$$

Main moment vector:

$$\overline{M} = \rho g \int_S \overline{r} \times \overline{n} z dS$$

Projections of the main force and moment vectors on the axes Oxyz:

$$P_x = \rho g \int_S z \cos(\bar{n}, \bar{i}) dS = \rho g \int_S z dS_x = \rho g z_{Cx} S_x$$

$$P_y = \rho g \int_S z \cos(\bar{n}, \bar{j}) dS = \rho g \int_S z dS_y = \rho g z_{Cy} S_y$$

$$P_z = \rho g \int_S z \cos(\bar{n}, \bar{k}) dS = \rho g \int_S z dS_z = \rho g V$$

$$M_x = \rho g \int_S z \left[y \cos(\bar{n}, \bar{k}) - z \cos(\bar{n}, \bar{j}) \right] dS = \rho g \int_S z \left(y dS_z - z dS_y \right)$$

$$M_y = \rho g \int_S z \left[z \cos(\bar{n}, \bar{i}) - x \cos(\bar{n}, \bar{k}) \right] dS = \rho g \int_S z \left(z dS_x - x dS_z \right)$$

$$M_z = \rho g \int_S z \left[x \cos(\bar{n}, \bar{j}) - y \cos(\bar{n}, \bar{i}) \right] dS = \rho g \int_S z \left(z dS_y - y dS_x \right)$$

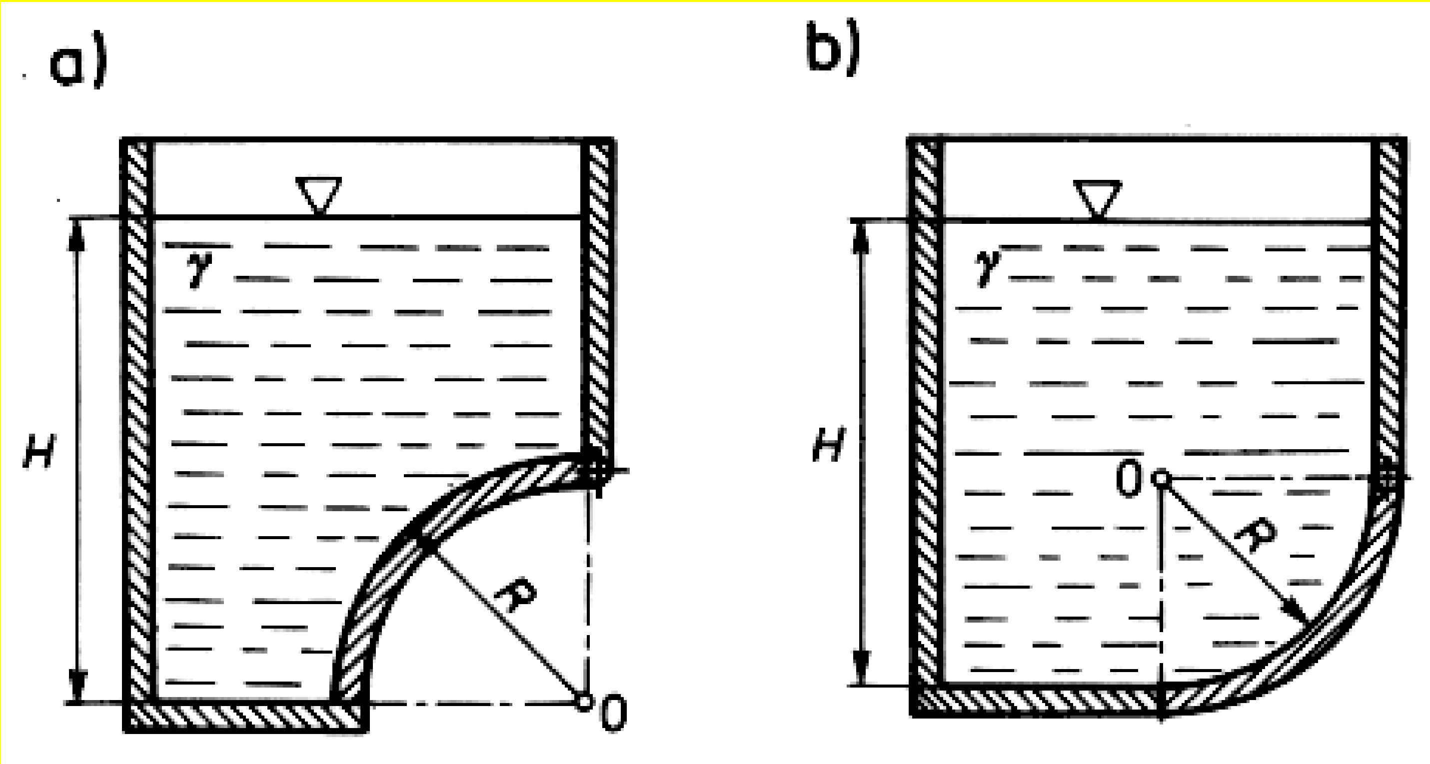
Conclusions

Projection of the hydrostatic force on an arbitrary horizontal direction is equal to the total hydrostatic force exerted on a flat wall, the area of which is equal to the projection of the curved wall onto the surface perpendicular to the considered direction.

As the areas of the horizontal projections do not depend on the shape of the wall *S*, but only on its contour, similarly the horizontal projections of the hydrostatic force depend only on the limiting contour of *S*.

The vertical projection of the hydrostatic force is equal to the weight of the prism of fluid contained between the wall S and its projection onto the free surface.

Example No. 3: a water tank is closed with a rotating flap shaped as a quarter of circular cylinder of radius R and length L. Determine the hydrostatic force acting on the flap in two cases a) and b). Assume water density equal to ρ.



Solution

Horizontal force components are equal in both cases and they are:

$$P_{Xa} = P_{Xb} = \rho g R L\left(H - \frac{R}{2}\right)$$

Vertical force components are respectively equal to:

$$P_{Za} = \rho g H R L - \rho g L \frac{\pi R^2}{4} = \rho g L R\left(H - \frac{\pi R}{4}\right)$$

$$P_{Zb} = \rho g H R L - \left(\rho g L R^2 - \rho g L \frac{\pi R^2}{4}\right) = \rho g R L\left(H - R + \frac{\pi R}{4}\right)$$

The resultant forces are respectively eual to:

$$P_a = \sqrt{P_{Xa}^2 + P_{Za}^2} \qquad P_b = \sqrt{P_{Xb}^2 + P_{Zb}^2}$$

They are inclined to the horizon by an angle: $\alpha = arctg \ \dfrac{P_X}{P_Z}$