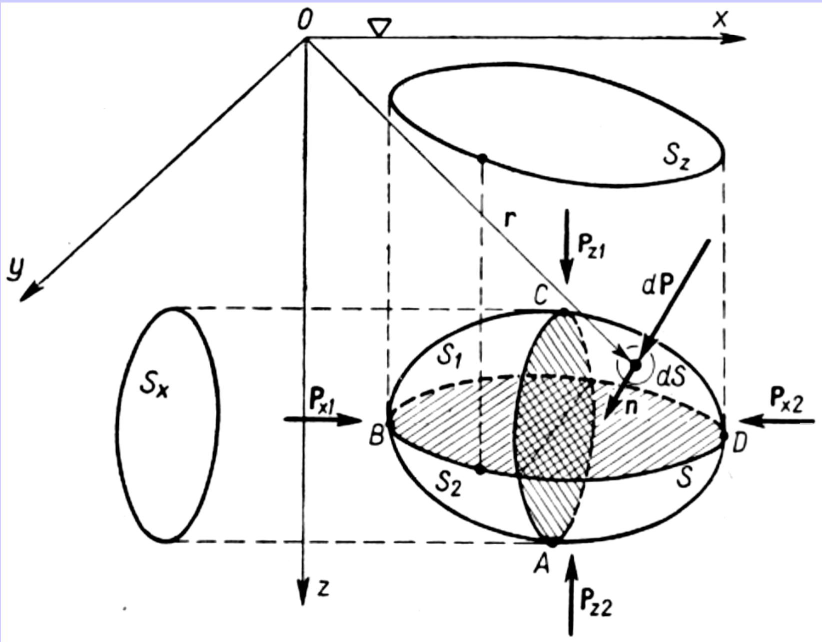
J. Szantyr – Lecture No. 5 – Floating of bodies

The Archimedes theorem

An elementary hydrostatic force $d\overline{P} = \overline{n}\rho gzdS$ acts on every element *dS*

The total hydrostatic force: $\overline{P} = \rho g \int_S \overline{n} zdS$

The main moment vector of this force: $\overline{M} = \rho g \int_S \overline{r} \times \overline{n} zdS$



The horizontal components of the hydrostatic force are equal zero. The total hydrostatic force is reduced to the vertical force acting on two parts of the suface joined at the common contour: the lower part BAD and the upper part BCD.

Force at the lower surface $$P_{z1} = \rho g \int_{S_1} z dS = \rho g V_1$$

Force at the upper surface $$P_{z2} = \rho g \int_{S_2} z dS = \rho g V_2$$

Resultant force $$P_z = P_{z1} - P_{z2} = -\rho g \left(V_2 - V_1 \right) = -\rho g V$$

Finally the buoyancy force $$W = -P_z = \rho g V$$

The hydrostatic buoyancy force acting on the body submerged in a liquid is equal to the weight of the liquid displaced by this body. The line of action of the buoyancy force passes through the centre of mass of the displaced liquid, or the centre of buoyancy.

Determination of the line of action of the buoyancy force.

Components of the main moment of the buoyancy force:

$$M_x = \rho g \int_S z\left(y dS_z - z dS_y\right) = \rho g \int_V \frac{\partial(yz)}{\partial z} dV - \rho g \int_V \frac{\partial z^2}{\partial y} dV = \rho g \int_V y dV = \rho g y_C V = y_C P_z$$

where: $y_C = \frac{1}{V} \int_V y dV$ is the y co-ordinate of the centre of volume V

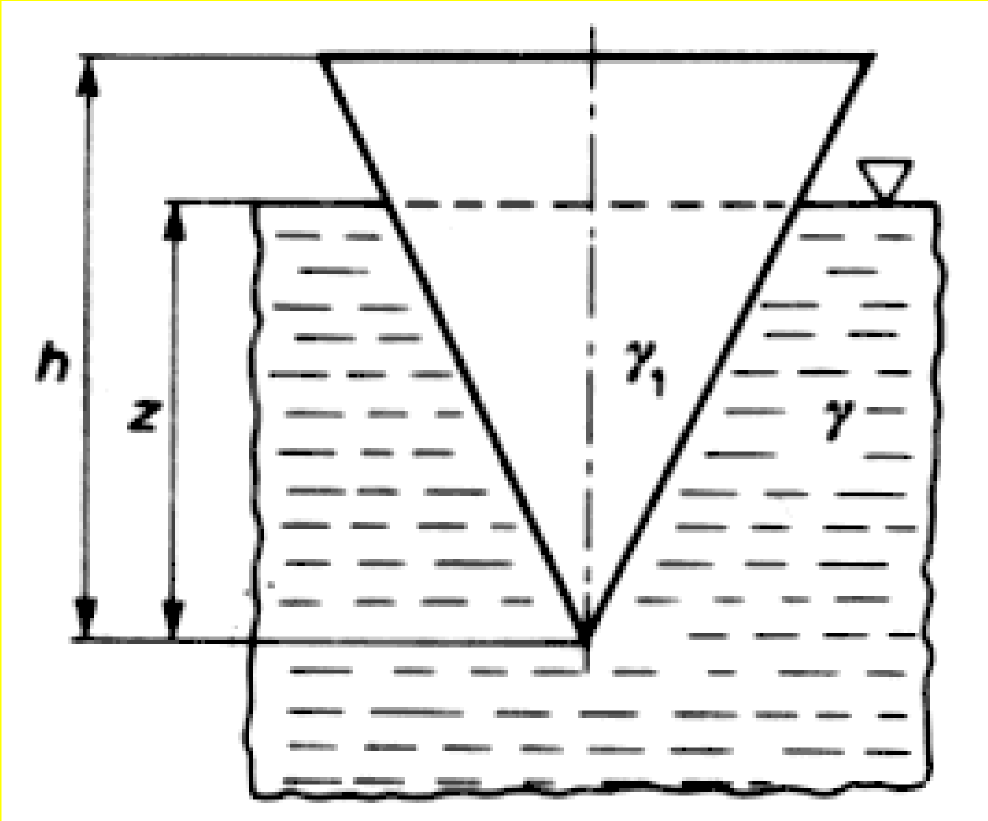
$$M_y = \rho g \int_S z\left(z dS_x - x dS_z\right) = \rho g \int_V \frac{\partial z^2}{\partial z} dV - \rho g \int_V \frac{\partial(xz)}{\partial z} dV = -\rho g \int_V x dV =$$

$$= -\rho g x_C V = -x_C P_z$$

where: $x_C = \frac{1}{V} \int_V y dV$ is the x co-ordinate of the centre of volume V

The line of action of the hydrostatic buoyancy force is directed vertically upwards and passes through the point $x_C y_C$

Example No. 1: A cube of height h made of material of specific weight γ_1 floats in a liquid with its vertex down. Determine the cube immersion if the specific weight of the liquid is equal to γ.



Solution

If the base area of the cube is A_h , and the waterline area is A_z , then the gravity force is equal to:

$$G = \frac{1}{3} A_h h \gamma_1$$

And the buoyancy force is: $W = \frac{1}{3} A_z z \gamma$

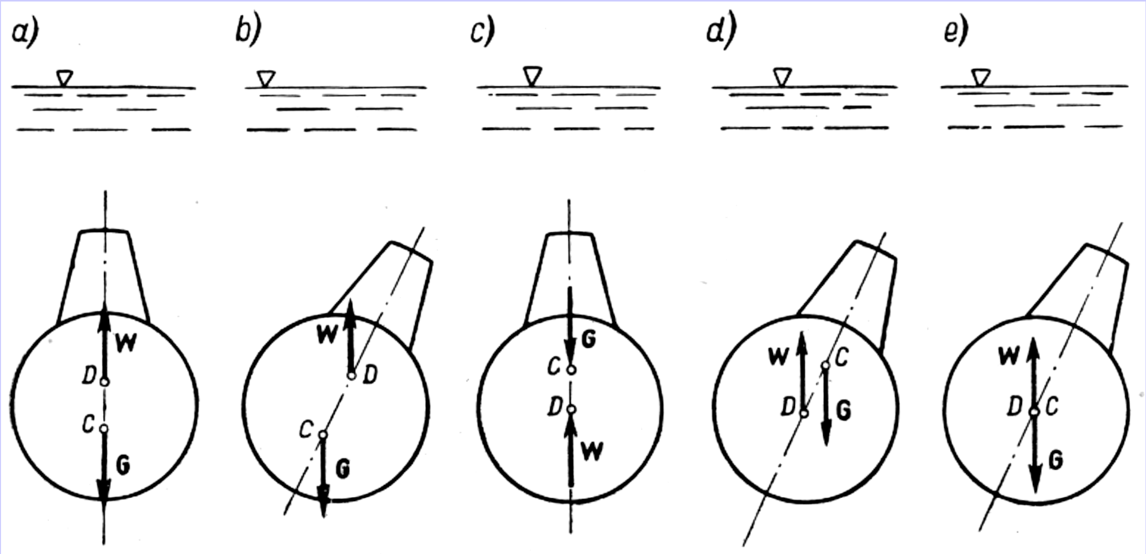
It follows from the equilibrium condition: $G = W \rightarrow \gamma_1 A_h h = \gamma A_z z \rightarrow z = h \frac{A_h}{A_z} \frac{\gamma_1}{\gamma}$

As there is: $\frac{A_h}{A_z} = \frac{h^2}{z^2}$ finally we have:

$$z = h \sqrt[3]{\frac{\gamma_1}{\gamma}}$$

Stability of the floating bodies

Stability of the completely submerged body



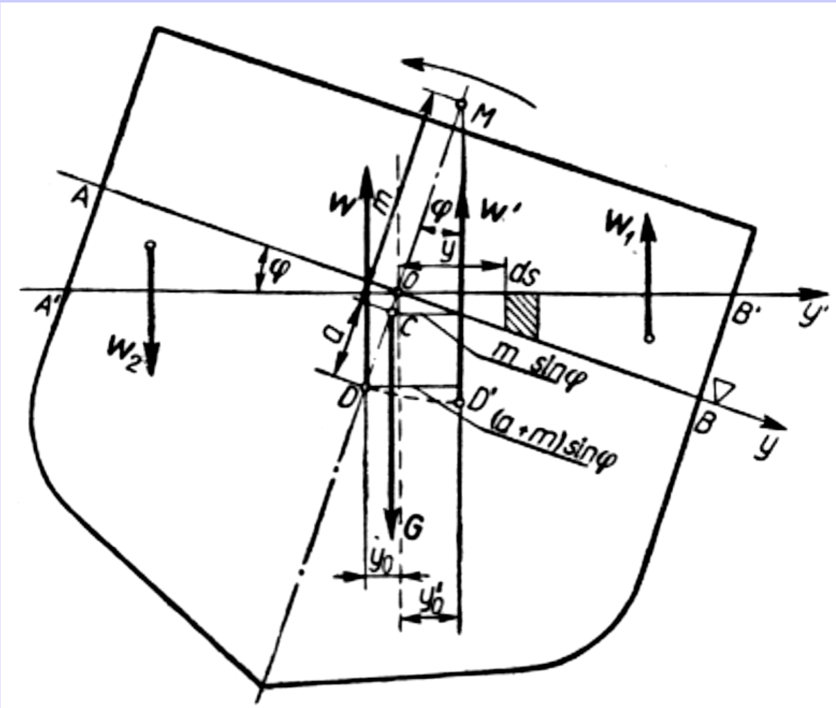
Stable equlibrium – Centre of buoyancy is located above the centre of gravity – Figs a) and b). Deflection from the equilibrium position generates the moment of the pair of forces acting against the deflection and enforcing return to the previous position.

Unstable equilibrium – centre of buoyancy is located below the centre of gravity – Figs c) and d). Deflection from the equilibrium position generates the moment of the pair of forces acting to increase the deflection.

Neutral equilibrium – Fig e) – in an arbitrary position of the body the forces of buoyancy and gravity balance each other without producing the moment which influences position of the body.

Conclusion: in the case of completely submerged body the stable equilibrium is ensured when the centre of buoyancy is located above the centre of gravity.

Stability of the partly submerged body



Assumption: the inclination angle is small

In the case of a partly submerged body the centre of buoyancy changes its location when inclined. The analysis of stability is based on determination of the position of the centre of buoyancy after the body is inclined by an angle φ.

The heeling moment: $M_0 = Ga\sin\varphi$

$G = \rho g V$ -weight of the body

$M_0 = \rho g a V \sin\varphi$

The righting moment

$dM_1 = y dW_1$ the right hand side of the inclined body

$$dW_1 = \rho g y dS \sin\varphi$$

$$dM_1 = \rho g y^2 dS \sin\varphi$$

$$M_1 = \rho g \sin\varphi \int_{S_1} y^2 dS \qquad M_2 = \rho g \sin\varphi \int_{S_2} y^2 dS$$

$$M = M_1 + M_2 = \rho g \sin\varphi \int_S y^2 dS = \rho g I_x \sin\varphi$$

The metacentric height *m* is defined (see the sketch):

$$M - M_0 = \rho g I_x \sin\varphi - \rho g a V \sin\varphi$$

$$m = \frac{I_x}{V} - a$$

I_x - moment of inertia of the waterline

$S = S_1 + S_2$ - area of the waterline

V – volume of the submerged part of the body

There are three possible cases:

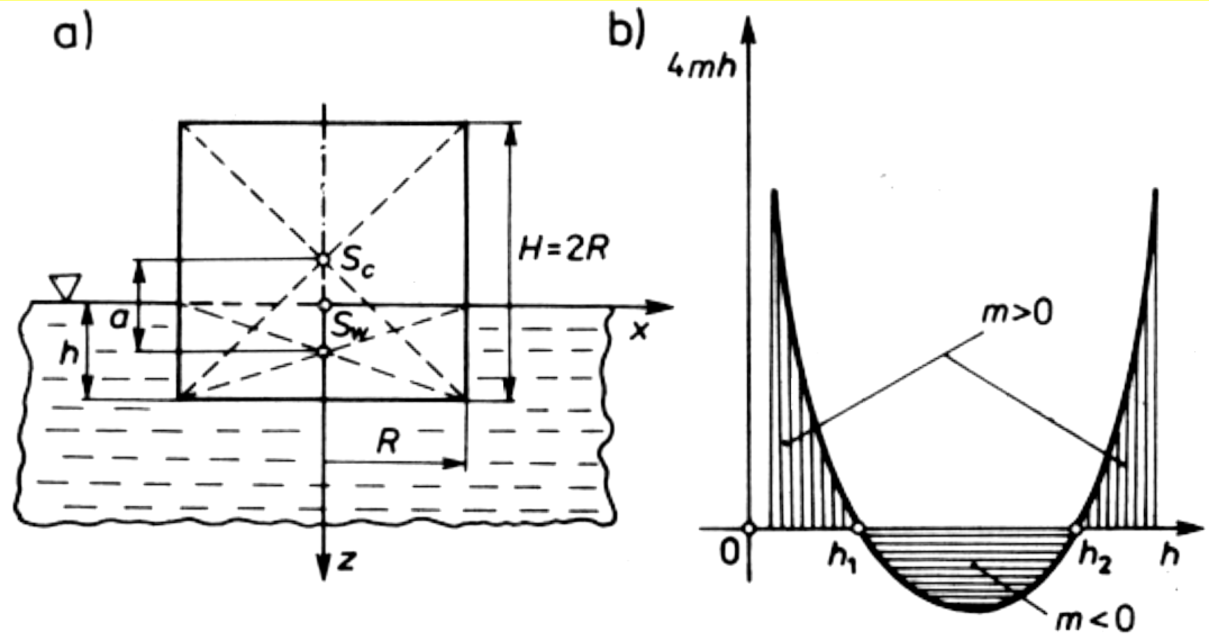
The metacentric height is positive – **stable equilibrium -** by deflection from the equilibrium position the righting moment of pair of forces is generated.

The metacentric height is equal zero – **neutral equilibrium**

The metacentric height is negative – **unstable equilibrium –** by deflection from the position of equilibrium the moment of the pair of forces acts to increase the deflection.

Conclusion: the partly submerged body may be in the state of stable equilibrium even if its centre of gravity is located above the centre of buoyancy. If the moment of inertia of the waterline is larger (the body is „wider”), the centre of gravity may be located higher.

Example No. 2



A circular cylinder having the base radius *R* and height *H=2R* floats in an upright position (Fig. a). The centre of gravity of the cylinder coincides with its geometrical centre. For which submergence of the cylinder *h* its equilibrium is stable?

The waterline of the cylinder is a circle, the inertia moment of which is:

$$I_x = \frac{\pi R^4}{4}$$

The volume of liquid displaced by the cylinder is:

$$V = \pi R^2 h$$

The distance between the centre of gravity of the cylinder and the centre of buoyancy is:

$$a = \frac{H}{2} - \frac{h}{2} = R - \frac{h}{2}$$

Substitution of the above relations to the formula for the metacentric height leads to:

$$m = \frac{I_x}{V} - a = \frac{R^2}{4h} - R + \frac{h}{2}$$

This may be transformed into:

$$4mh = R^2 - 4Rh + 2h^2$$

This function may be plotted as a parabola (Fig. b), the zero points of which may be determined by equating the right hand side to zero:

$$2h^2 - 4Rh + R^2 = 0$$

The above quadratic equation has two solutions:

$$h_1 = \frac{2-\sqrt{2}}{2} R \approx 0{,}29R \qquad h_2 = \frac{2+\sqrt{2}}{2} R \approx 1{,}7R$$

The diagram in Fig. B shows, that the stable equilibrium of the cylinder is achieved at the shallow draught according to the condition:

$$h < 0,29R$$

and at the deep draught:

$$h > 1,7R$$

For the intermediate draughts of the cylinder its equilibrium is unstable.