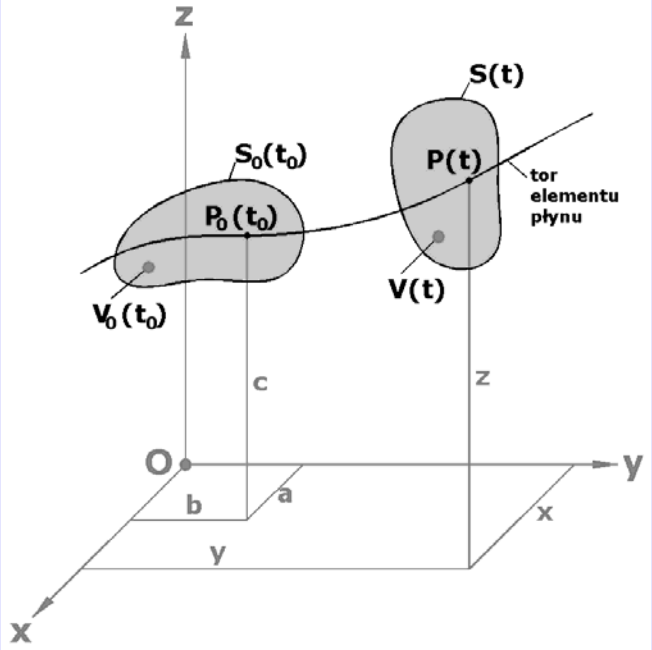
J. Szantyr – Lecture No. 6: Kinematics of fluids – Lagrange and Euler approach – Stream lines – Paths of fluid elements

Lagrange approach is based on description of the spatial motion of a certain separated mass of fluid composed always of the same molecules.



V – volume of a certain mass of fluid (fluid volume) surrounded by the surface *S*, which is <u>impenetrable</u> for the fluid elements.

The mass of fluid moves from the location V_0 at time t_0 to the location V in time t.

The fluid element *P*, constituting a part of the volume *V*, moves in space tracing the path, which may be described by the following set of parametric equations with time *t* as the parameter:

$$x = x(a,b,c,t)$$

$$y = y(a,b,c,t)$$

$$z = z(a,b,c,t)$$

<u>Changing the quantities *a*, *b* and *c* in the equations we describe different fluid elements.</u>

The parameters describing fluid motion depend in the same way on *a*, *b*, *c*, *t*:

$$\bar{u} = \bar{u}(a,b,c,t)$$

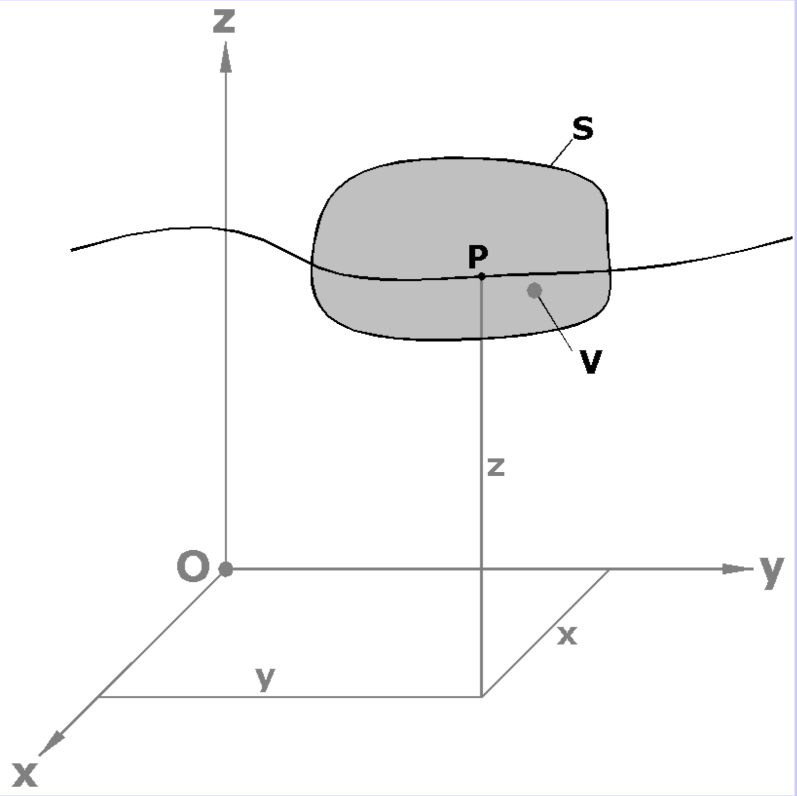
$$p = p(a,b,c,t)$$

$$\rho = \rho(a,b,c,t)$$

where: $\bar{u} = \bar{i}u_x + \bar{j}u_y + \bar{k}u_z$

$$u_x = \frac{dx}{dt} \qquad u_y = \frac{dy}{dt} \qquad u_z = \frac{dz}{dt}$$

Euler method is based on selection of the stationary control volume *V* surrounded by the control surface *S*. Different fluid elements flow through this volume (crossing the surface *S*) with different values of velocity, pressure, density etc. The values of these quantities in different points of the control volume are the subject of the description.



$$\bar{u} = \bar{u}(x, y, z, t)$$

$$p = p(x, y, z, t)$$

$$\rho = \rho(x, y, z, t)$$

where:

$$\bar{u} = \bar{i}\,u_x(x, y, z, t) + \bar{j}\,u_y(x, y, z, t) + \bar{k}\,u_z(x, y, z, t)$$

The material derivative

The material derivative is a particular interpretation of the complete derivative of a function of several variables, related to the Eulerian description of the fluid motion. It shows, how an arbitrary flow parameter describing the fluid element changes with time when the element is moving in the field of this parameter. It is explained below using the example of an arbitrary scalar parameter *H*, which is a direct and involved function of time. If *H* is the function of Euler variables, then we have:

$$H = H\big(t, x(t), y(t), z(t)\big)$$

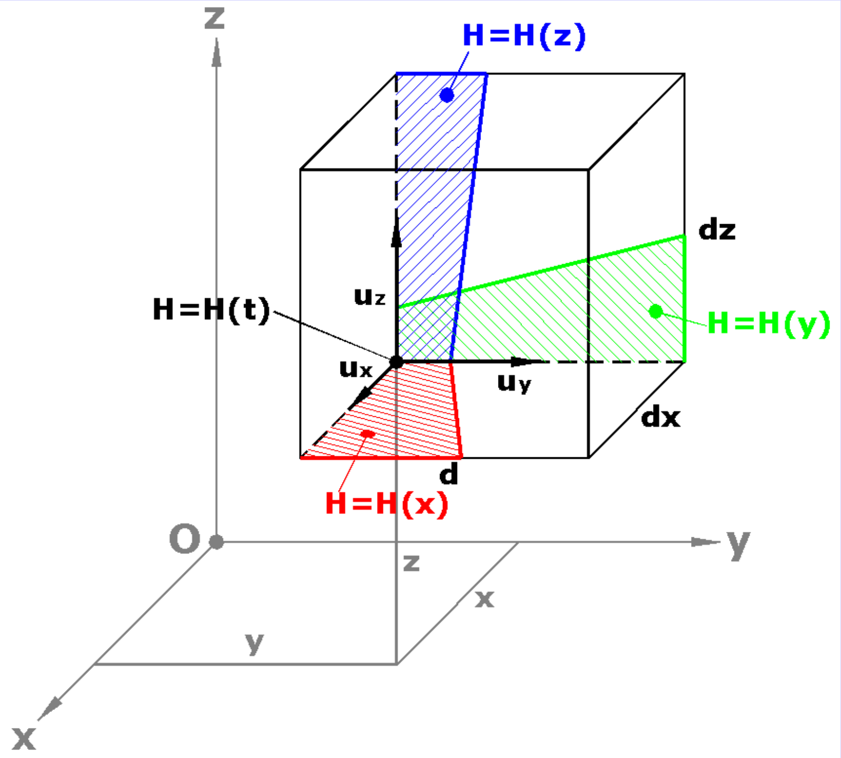
Following the definition of the complete differential we have:

$$\frac{DH}{Dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial y}\frac{dy}{dt} + \frac{\partial H}{\partial z}\frac{dz}{dt}$$

but: $\dfrac{dx}{dt} = u_x$ $\dfrac{dy}{dt} = u_y$ $\dfrac{dz}{dt} = u_z$ what leads to:

$$\frac{DH}{Dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x}u_x + \frac{\partial H}{\partial y}u_y + \frac{\partial H}{\partial z}u_z = \frac{\partial H}{\partial t} + \bar{u} \bullet \nabla H = \frac{\partial H}{\partial t} + \bar{u} \bullet gradH$$

Material derivative=local derivative+convection derivative



The local derivative shows the change of the parameter *H* with time in the point *(x, y, z)*, resulting from the unsteadiness of the field *H*.

The convective derivative shows the change of the parameter *H* with time, resulting from the motion of the fluid element with velocity *u* from the point of one value of *H* to the point of another value of *H*.

Application of the material derivative operator to the components of the velocity field enables calculation of the material acceleration, i.e. the acceleration of the fluid element moving in the unsteady and non-uniform field of flow.

$$\frac{Du_x}{Dt} = \frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} = a_x$$

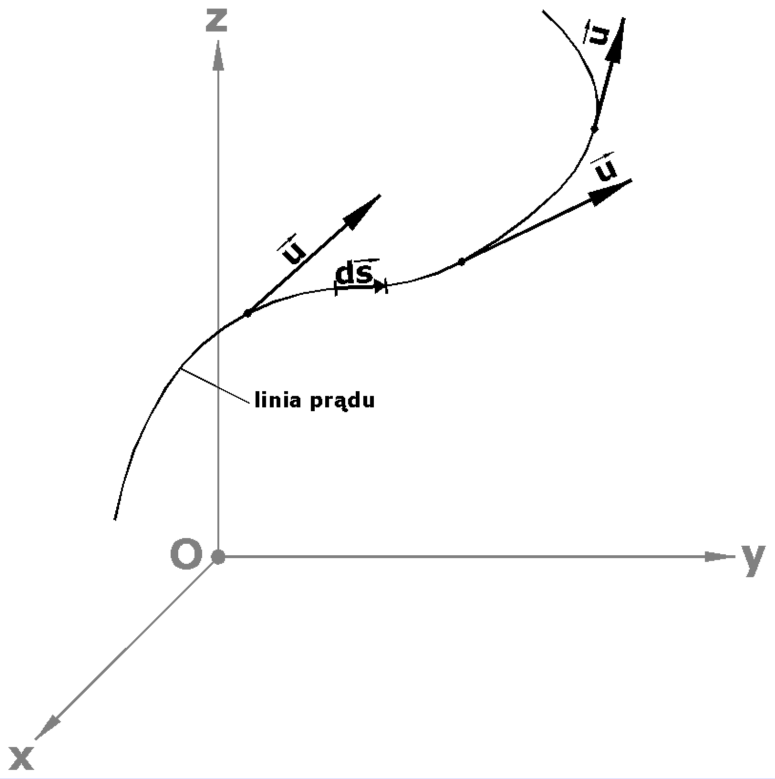
$$\frac{Du_y}{Dt} = \frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} + u_z \frac{\partial u_y}{\partial z} = a_y$$

$$\frac{Du_z}{Dt} = \frac{\partial u_z}{\partial t} + u_x \frac{\partial u_z}{\partial x} + u_y \frac{\partial u_z}{\partial y} + u_z \frac{\partial u_z}{\partial z} = a_z$$

or in vector notation:

$$\frac{D\overline{u}}{Dt} = \frac{\partial \overline{u}}{\partial t} + \overline{u} \bullet grad\overline{u} = \frac{\partial \overline{u}}{\partial t} + \left(\overline{u}\nabla\right)\overline{u}$$

Stream line is the line of the vector field of velocity, i.e. the line tangent to the velocity vectors in every point of the velocity field in the given instant of time. If *ds* is the element of the stream line, and *u* – the velocity vector, then we have:



$$d\overline{s} \times \overline{u} = 0$$ condition of tangentiality

or:

$$u_z dy - u_y dz = 0$$

$$u_x dz - u_z dx = 0$$

$$u_y dx - u_x dy = 0$$

what leads to the stream line equation:

$$\frac{dx}{u_x} = \frac{dy}{u_y} = \frac{dx}{u_z}$$

In general only one, univocally determined, stream line passes through every point of the velocity field. If more stream lines converge in one point of the field then this is a **singular point**. If we draw stream lines through a line not being a stream line, we obtain a **stream surface**. If this line is a closed curve, we obtain a **stream tube**. If this tube has an infinitesimal cross-section, we obtain a **stream filament**. Stream tube is a good model of a pipeline, for which we may determine:

volumetric intensity of flow: $$Q = \int_S u_n dS$$

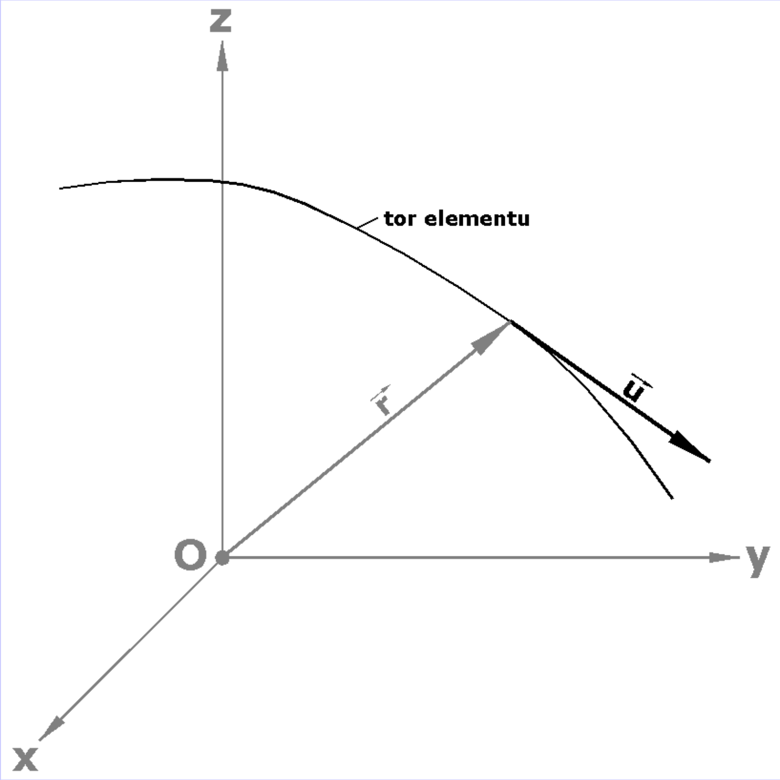
volumetric mean velocity: $$\tilde{u} = \frac{1}{S}\int_S u_n dS$$

mass intensity of flow: $$M = \int_S \rho u_n dS$$

mass mean velocity: $$\tilde{u} = \frac{\int_S \rho u_n dS}{\int_S \rho dS}$$

where: u_n is the velocity component normal to the cros-section S

Path of the fluid element or **trajectory** is the geometrical location of the points in the field of flow through which the element passes in the consecutive instants of time.



Vector equation of the path:

$$\frac{d\bar{r}}{dt} = \bar{u}(\bar{r}, t)$$

In the scalar form:

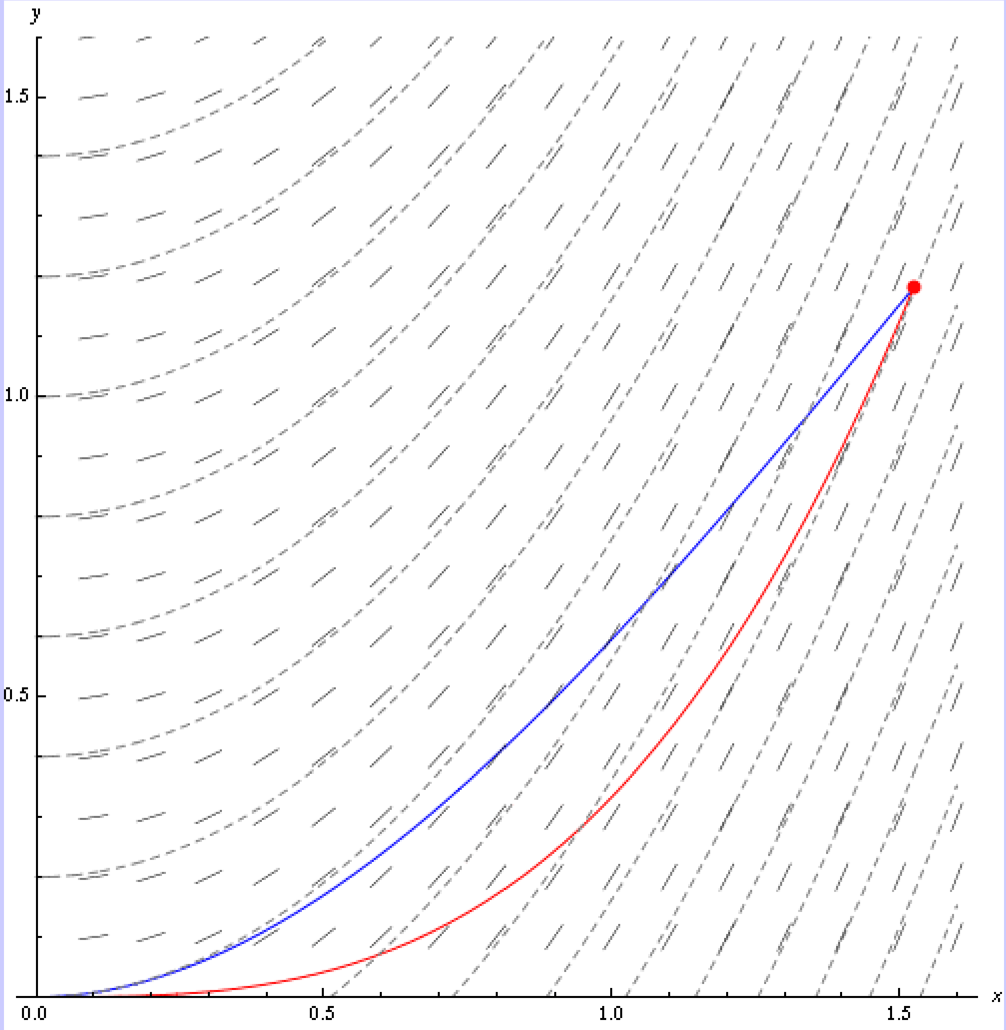
$$\frac{dx}{dt} = u_x(x, y, z, t) \qquad \frac{dy}{dt} = u_y(x, y, z, t)$$

$$\frac{dz}{dt} = u_z(x, y, z, t)$$

Solution requires taking into account the initial conditions for $t = t_0$

$$x(t) = x_0 \qquad y(t) = y_0 \qquad z(t) = z_0$$

In an unsteady flow the stream lines, paths of the elements and streak lines **do not coincide.**



Stream lines – grey colour

Paths of the elements – red colour

Streak lines – blue colour

Streak line is the trace of the fluid element drifting in the unsteady velocity field of the moving fluid.