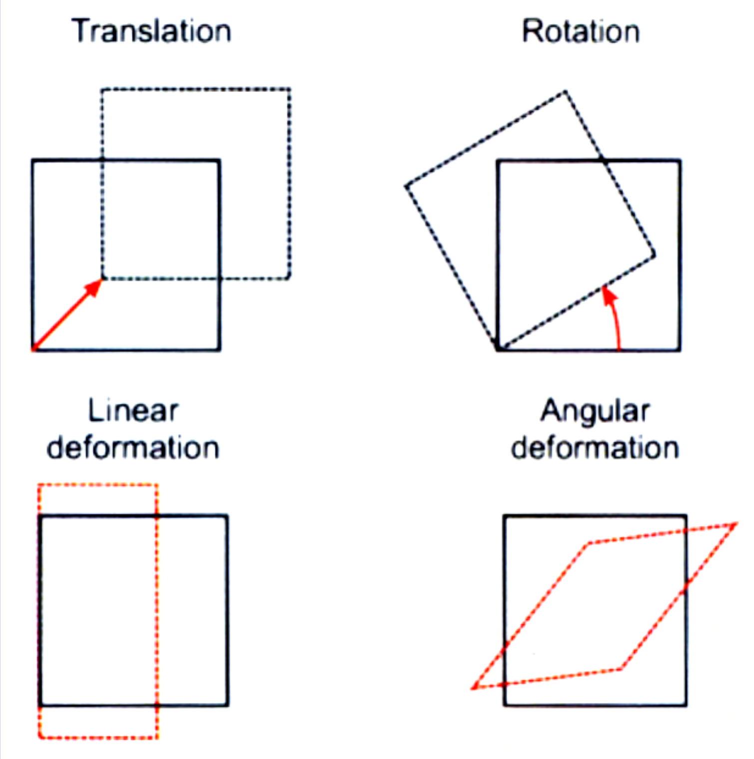
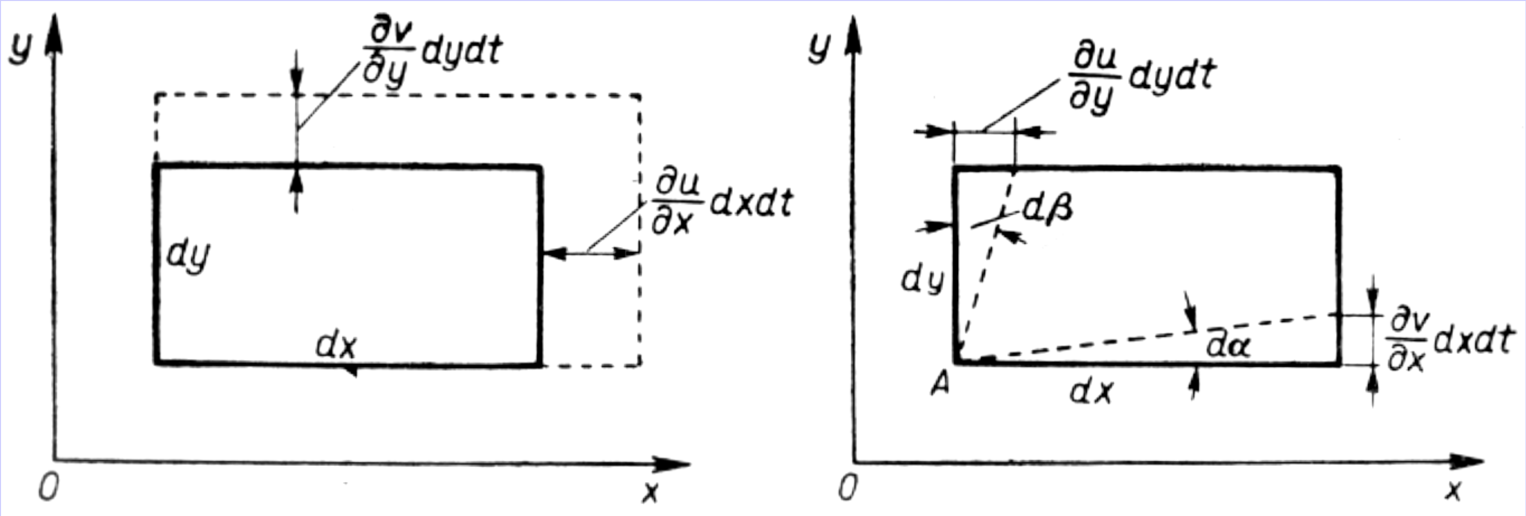
J. Szantyr – Lecture No. 7 – A general motion of the fluid element

A general motion of the rigid body may be considered as a sum of linear translation and rotation. As the fluids are not rigid, in their motion the deformation of the fluid elements must be additionally considered.



Thus the general motion of the fluid element may be treated as the superposition of the linear translation, rotation around the temporary centre and deformation, which in turn may be divided into linear deformation and angular (shearing) deformation.

Deformations in the two-dimensional case



Velocity of the fluid motion is: $\bar{u} = \bar{i}u + \bar{j}v$

The linear deformation of the fluid element takes place when the velocity component *u* varies in direction *x* and/or the velocity component *v* varies in direction *y* (left side of the picture). This may lead to the change of the element volume in the time *dt* by the value:

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dxdydt$$

Where the quantities in parantheses are the linear deformation velocities:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y}$$

The angular (shearing) deformation of the fluid element takes place when the velocity component u varies in the direction y and/or the velocity component v varies in the direction x (right hand side of the picture). This leads to the rotation of the element walls by the angles:

$$d\alpha = \frac{\partial v}{\partial x} dt \qquad d\beta = \frac{\partial u}{\partial y} dt$$

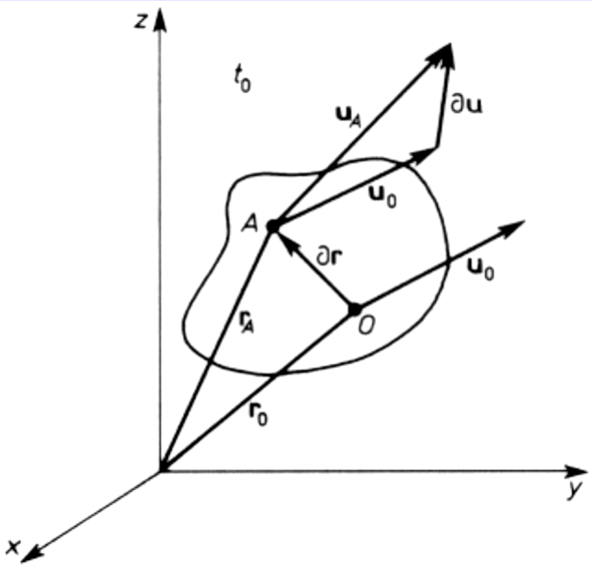
The measure of the combined angular deformation is the expression:

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$

The rigid rotation of the fluid element may be regarded as the sum of two deformations selected in such a way that the angles between the element walls remain the right angles. The angular velocity of such rotation may be written as:

$$\Omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

Deformations in the three-dimensional case



The fluid element performs an arbitrary motion composed of translation with velocity $\overline{u}_0$, rotation around the centre O and deformation. Due to the rotation and deformation the vector $\partial\overline{r}$ linking point A with the centre changes. In a general case this vector changes its module and its direction. It may be written:

$$d\left(\partial\overline{r}\right)=\left(\overline{u}_A-\overline{u}_0\right)dt$$

If the distance between the points *O* and *A* is assumed to be small, the difference of their velocities may be developed into a Taylor series and only the first term of this development may be taken into account:

$$\overline{u}_A=\overline{u}_0+\nabla\overline{u}_0\cdot\left(\overline{r}_A-\overline{r}_0\right)+\frac{1}{2}\nabla^2u_0\cdot\left(\overline{r}_A-\overline{r}_0\right)^2......$$

hence:

$$\partial\overline{u} = \overline{u}_A - \overline{u}_0 = \frac{\partial(\delta\overline{r})}{\partial t} = \nabla\overline{u}_0 \cdot \partial\overline{r}$$

where: $\nabla\overline{u}_0$ is the tensor of relative velocity of the point *A* with respect to the centre *O*

$$\nabla\overline{u} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Where the velocity vector has the form:

$$\overline{u} = \overline{i}u + \overline{j}v + \overline{k}w$$

The tensor of relative velocity may be considered as the sum of two tensors: anti-symmetric and symmetric. The anti-symmetric tensor describes the rotation of the fluid element as a rigid body. Its terms represent the components of the angular velocity vector ω:

$$[\Omega] = \begin{vmatrix} 0, -\omega_z, \omega_y \\ \omega_z, 0, -\omega_x \\ -\omega_y, \omega_x, 0 \end{vmatrix}$$

where:

$$\overline{\omega} = \overline{i}\omega_x + \overline{j}\omega_y + \overline{k}\omega_z$$

$$\overline{\omega} = \frac{1}{2} rot\overline{u}$$

The respective terms of the tensor are described by the following expressions:

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial x}\right) \qquad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \qquad \omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

The symmetric tensor describes the deformation of the fluid element and it is known as the rate of deformation tensor:

$$[D] = \begin{vmatrix} \varepsilon_{xx}, \varepsilon_{xy}, \varepsilon_{xz} \\ \varepsilon_{yx}, \varepsilon_{yy}, \varepsilon_{yz} \\ \varepsilon_{zx}, \varepsilon_{zy}, \varepsilon_{zz} \end{vmatrix}$$

Where the respective terms are described by the expressions:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \varepsilon_{yz} = \varepsilon_{zy} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} \qquad \varepsilon_{xz} = \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

Finally, the general motion of the fluid element may be described by the following relation:

$$\overline{u}_A = \overline{u}_0 + \overline{\omega}_0 \times \partial\overline{r} + \left[D\right]_0 \cdot \partial\overline{r}$$

The first Helmholtz theorem

The velocity of an arbitrary point of the fluid element is composed of:

-translation velocity of the point selected as the centre,

-rotation velocity around the axis passing through this centre (the vector of this velocity defines the axis of rotation),

-deformation velocity of the fluid element.

In comparison with the analogical motion of the rigid body the following differences may be noticed:

-the formula for the fluid is valid only close to the rotation centre,

-in the fluid there is an additional velocity of deformation.

Rotational motion of the fluid

The flow is rotational if everywhere or almost everywhere (except the finite number of points, lines or surfaces) the rotation of the velocity field is not equal to zero. Then the vector of rotation (or vorticity vector) may be ascribed to every or almost every point in space:

$$\overline{\Omega} = rot\bar{u} = 2\overline{\omega}$$

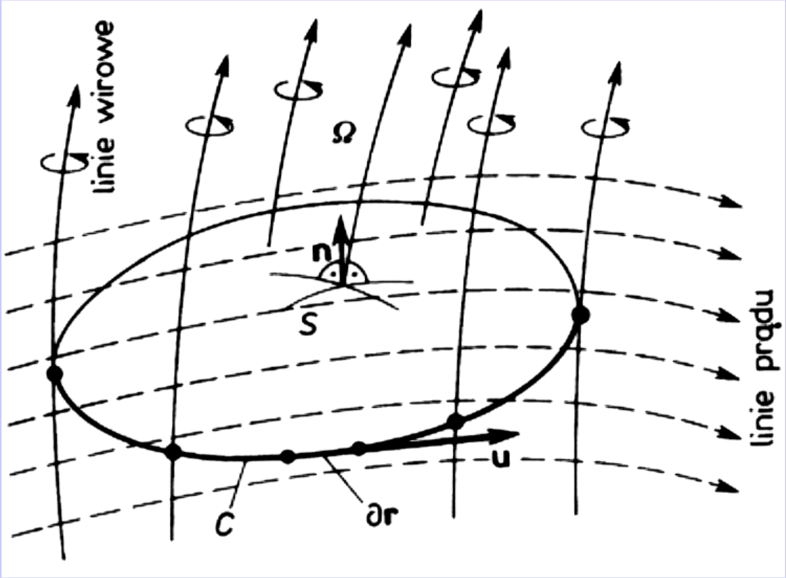
The components of the vorticity vector are defined as:

$$\Omega_x = 2\omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}$$

$$\Omega_y = 2\omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$$

$$\Omega_z = 2\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Through the analogy to the stream lines the vortex lines may be determined as the lines of vector field of vorticity, i.e. the lines tangent to the vorticity vectors in every point of space.



Equation of the vortex line:

$$\frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z}$$

Circulation of the velocity vector is defined as:

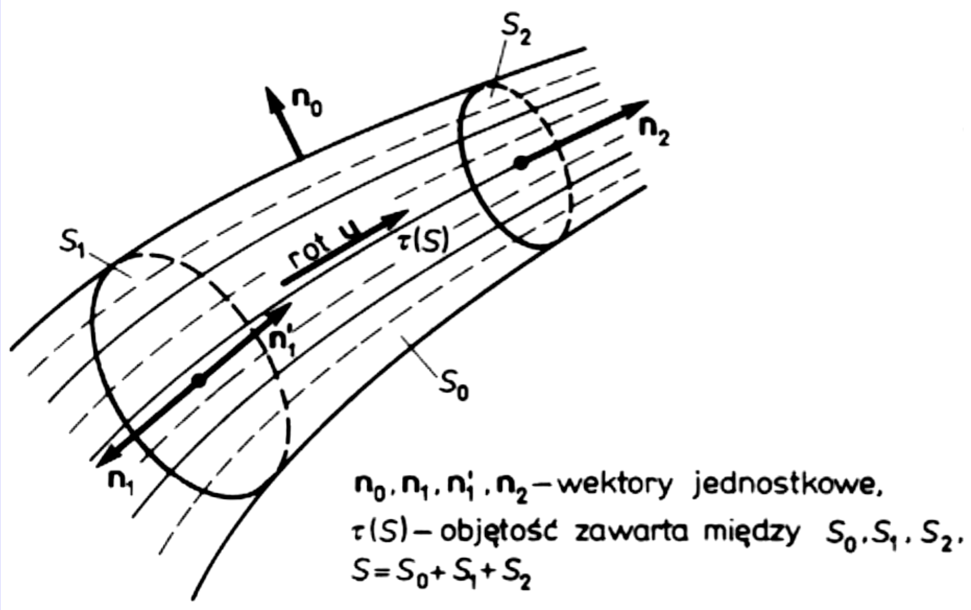
$$\Gamma = \oint_C \bar{u} \bullet d\bar{r} = \oint_C (udx + vdy + wdz) = \int_S rot\bar{u} \bullet dS$$

Stokes theorem: Circulation of the velocity around an arbitrary contour C is equal to the stream of vorticity passing through an arbitrary surface based on this contour.

Vortex lines passing through a line which is not a vortex line form the **vortex surface**. If this line is a closed curve, they form a **vortex tube.** The vortex tube of an infinitesimal diameter is a **vortex filament.**

Thomson theorem: in the flow of an ideal barotropic fluid under the action of the potential field of mass forces the circulation of velocity around an arbitrary closed fluid contour does not change with time.

Second theorem of Helmholtz: in the flow of an ideal barotropic fluid under the action of the potential field of mass forces the intensity of the vortex filament does not change along its length and it is constant in time.



Conclusions:

-a vortex filament may not disappear or be generated in the fluid,

-a vortex filament may form a closed curve,

-a vortex filament may terminate at the free surface or at the rigid body surface,

-the same fluid elements participate in vortex motion at all times.

In practical modelling the flow may be divided into the rotational flow region and irrotational flow region. Both these regions are mutually interdependent. The rotational region may be modelled by vortex filaments. In such a case it is important to determine the velocity field generated by the vorticity field, i.e. the operation opposite to the calculation of rotation of the velocity field.

The Biot-Savart formula

$$d\overline{V} = \frac{\Gamma}{4\pi}\frac{d\overline{s}\times\overline{r}}{r^3}$$

$$\overline{V} = \frac{\Gamma}{4\pi}\int_L \frac{d\overline{s}\times\overline{r}}{r^3}$$