Principle of mass conservation: in a closed physical system mass cannot be generated or annihilated.

Assumptions:

-we consider an unsteady three-dimensional flow of a compressible fluid,

-the fluid fills the space in a continuous way (no bubbles etc.),

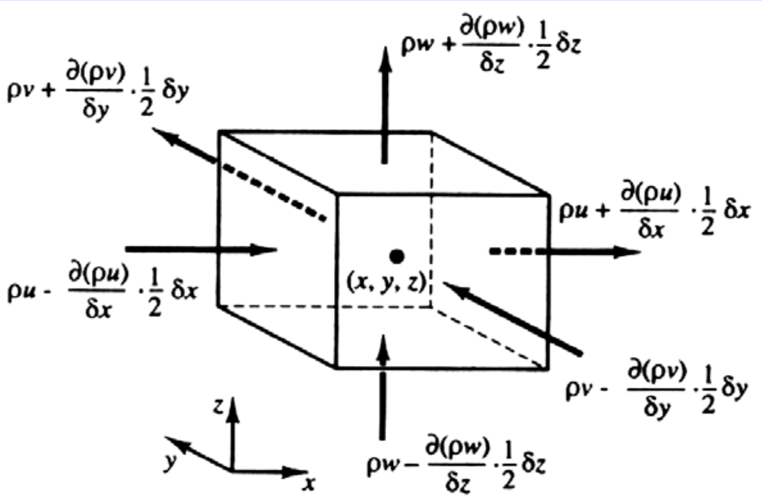
-we apply the Eulerian approach – a stationary control volume surrounded by a control surface.

With these assumptions the mass conservation principle reads:

the change of mass in the volume = the flow of mass through the surface

The change of mass in the control volume is:

$$\frac{\partial}{\partial t}(\rho \delta x \delta y \delta z) = \frac{\partial \rho}{\partial t} \delta x \delta y \delta z$$



In turn the flow through the control surface is:

$$\left(\rho u - \frac{\partial(\rho u)}{\partial x}\frac{1}{2}\delta x\right)\delta y\delta z - \left(\rho u + \frac{\partial(\rho u)}{\partial x}\frac{1}{2}\delta x\right)\delta y\delta z +$$

$$+\left(\rho v - \frac{\partial(\rho v)}{\partial y}\frac{1}{2}\delta y\right)\delta x\delta z - \left(\rho v + \frac{\partial(\rho v)}{\partial y}\frac{1}{2}\delta y\right)\delta x\delta z +$$

$$+\left(\rho w - \frac{\partial(\rho w)}{\partial z}\frac{1}{2}\delta z\right)\delta x\delta y - \left(\rho w + \frac{\partial(\rho w)}{\partial z}\frac{1}{2}\delta z\right)\delta x\delta y$$

Equating of both above expressions leads to (after dividing both sides by the control volume):

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = \frac{\partial \rho}{\partial t} + div(\rho \bar{u}) = 0$$

In the case of the steady flow of the compressible fluid the mass conservation equation takes the form:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = div(\rho \bar{u}) = 0$$

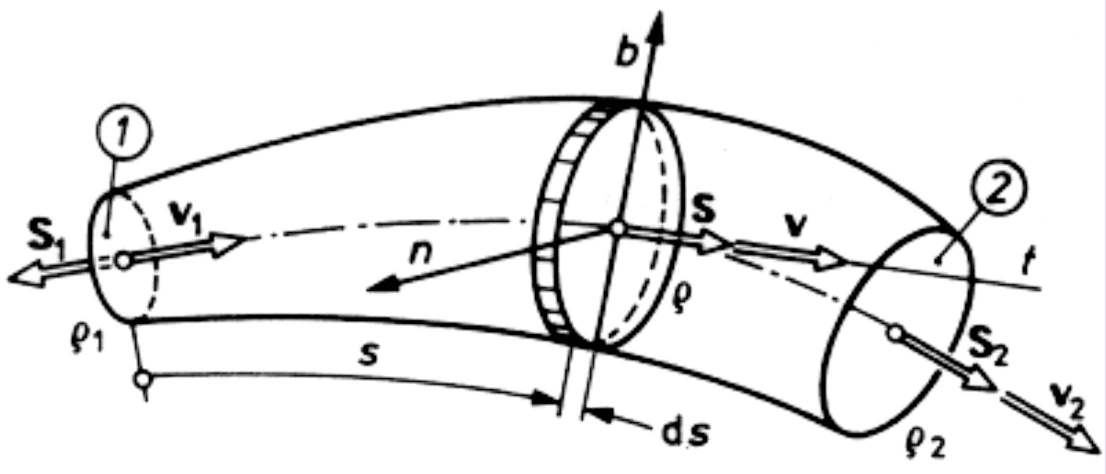
In the case of the steady flow of the incompressible fluid the mass conservation equation takes the form:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = div \bar{u} = 0$$

In the case of the moving fluid element (Lagrangian approach) the mass conservation equation takes the form:

$$\frac{\partial \rho}{\partial t} + div(\rho \bar{u}) = \frac{\partial \rho}{\partial t} + \bar{u} \cdot grad \rho + \rho div \bar{u} = \frac{D\rho}{Dt} + \rho div \bar{u}$$

Mass conservation equation for a stream tube



The difference of mass flow between the cross-sections 1 and 2:

$$\int_{S_1} \rho(s,t)u(s,t)dS - \int_{S_2} \rho(s,t)u(s,t)dS = \int_2^1 \frac{\partial}{\partial s}\left(\tilde{\rho}\tilde{u}S\right)ds$$

The corresponding average values are defined as:

$$\tilde{\rho}_1 = \frac{1}{S_1}\int_{S_1} \rho_1 dS_1 \qquad \tilde{\rho}_2 = \frac{1}{S_2}\int_{S_2} \rho_2 dS_2$$

$$\tilde{u}_1 = \frac{1}{S_1\tilde{\rho}_1}\int_{S_1} \rho_1 u_1 dS_1 \qquad \tilde{u}_2 = \frac{1}{S_2\tilde{\rho}_2}\int_{S_2} \rho_2 u_2 dS_2$$

This difference may be generated due to the compression or expansion of the fluid in the control volume limited by the cross-sections 1 and 2. In such a case the rate of change of mass in the control volume is:

$$\frac{\partial}{\partial t}\int_1^2 \tilde{\rho}(s,t)S(s,t)ds$$

According to the mass conservation principle the rate of change of mass in the control volume must be equal to the flow of mass through the control surface.

$$\frac{\partial}{\partial t}\int_1^2 \tilde{\rho}Sds = -\int_1^2 \frac{\partial(\tilde{\rho}\tilde{u}S)}{\partial s}ds \qquad \text{or:} \qquad \int_1^2 \left[\frac{\partial(\tilde{\rho}S)}{\partial t} + \frac{\partial(\tilde{\rho}\tilde{u}S)}{\partial s}\right]ds = 0$$

As the above equation is valid for an arbitrarily selected cross-sections 1 and 2, the function under the integral should be equal to zero. This leads to the following mass conservation equation:

$$\frac{\partial(\tilde{\rho}S)}{\partial t} + \frac{\partial(\tilde{\rho}\tilde{u}S)}{\partial s} = 0$$

For an incompressible fluid (constant density) we obtain:

$$\frac{\partial S}{\partial t} + \frac{\partial(\tilde{u}S)}{\partial s} = 0$$

In turn, for a steady flow we obtain:

$$\frac{\partial(\tilde{\rho}\tilde{u}S)}{\partial s} = 0 \qquad \text{or:} \qquad \tilde{\rho}\tilde{u}S = const$$

Conclusions:

-in the steady flow of compressible fluid the mass intensity of flow (the stream of mass of the fluid) through any cross-section of a stream tube is constant,

-in the steady flow of an incompressible fluid the volumetric intensity of flow (the stream of volume of the fluid) is constant and velocity is inversely proportional to the area of the stream tube cross-section.