J. Szantyr – Lecture No. 9 – Momentum conservation equation 1

The second law of Newton: the rate of change of momentum of a fluid element is equal to the sum of forces acting on this element.

$$\frac{D(m\overline{u})}{Dt} = \sum \overline{F}$$

The rate of change of momentum of the fluid element (the left hand side) is defined by the material derivative of its velocity:

$$\rho\frac{Du}{Dt} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \frac{\partial(\rho u)}{\partial t} + div(\rho u\overline{u})$$

$$\rho\frac{Dv}{Dt} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \frac{\partial(\rho v)}{\partial t} + div(\rho v\overline{u})$$

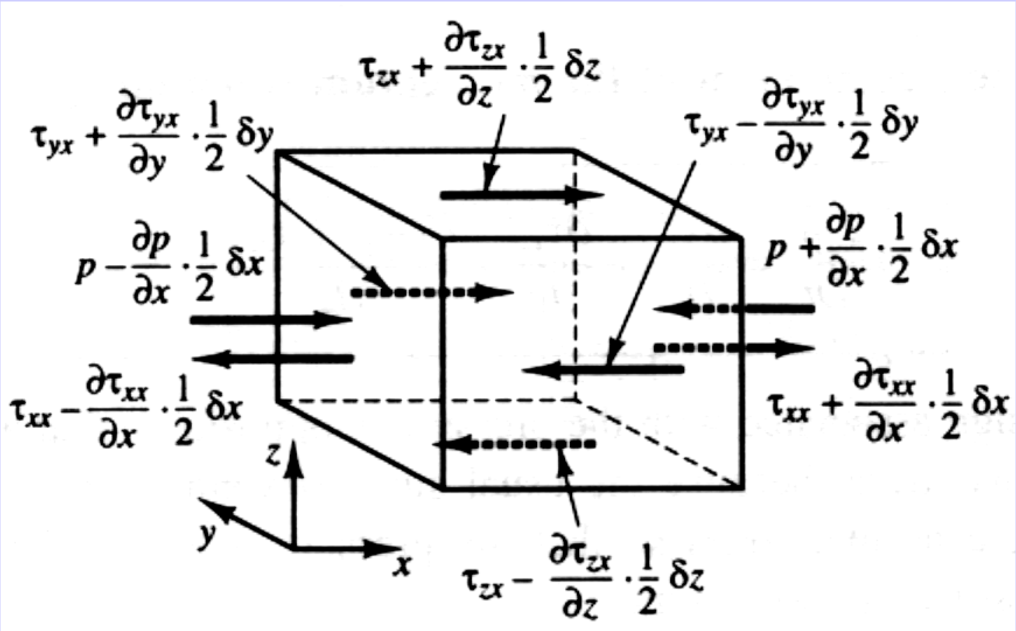
$$\rho\frac{Dw}{Dt} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \frac{\partial(\rho w)}{\partial t} + div(\rho w\overline{u})$$

The right hand side is composed of the two categories of forces:

-surface forces (pressure forces and viscosity forces),

-mass forces (gravity forces, Coriolis forces, electromagnetic forces)

For example we will formulate the complete equation for the x direction, using the system of surface forces as in the picture:



Forces acting on the element walls perpendicular to the x direction

$$\left[\left(p-\frac{\partial p}{\partial x}\frac{1}{2}\delta x\right)-\left(\tau_{xx}-\frac{\partial \tau_{xx}}{\partial x}\frac{1}{2}\delta x\right)\right]\delta y\delta z+\left[-\left(p+\frac{\partial p}{\partial x}\frac{1}{2}\delta x\right)+\left(\tau_{xx}+\frac{\partial \tau_{xx}}{\partial x}\frac{1}{2}\delta x\right)\right]\delta y\delta z=$$

$$=\left(-\frac{\partial p}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}\right)\delta x\delta y\partial z$$

Forces acting on the element walls perpendicular to the y direction

$$-\left(\tau_{yx}-\frac{\partial \tau_{yx}}{\partial y}\frac{1}{2}\delta y\right)\delta x\delta z+\left(\tau_{yx}+\frac{\partial \tau_{yx}}{\partial y}\frac{1}{2}\delta y\right)\delta x\delta z=\frac{\partial \tau_{yx}}{\partial y}\delta x\delta y\delta z$$

Forces acting on the element walls perpendicular to the z direction

$$-\left(\tau_{zx}-\frac{\partial \tau_{zx}}{\partial z}\frac{1}{2}\delta z\right)\delta x\delta y+\left(\tau_{zx}+\frac{\partial \tau_{zx}}{\partial z}\frac{1}{2}\delta z\right)\delta x\delta y=\frac{\partial \tau_{zx}}{\partial z}\delta x\delta y\delta z$$

After adding the above expressions together and dividing by the element volume we obtain the surface forces acting in direction x:

$$\frac{\partial(-p+\tau_{xx})}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}$$

After supplementing the expression with the unit mass force f and substituting it to the initial formula we obtain:

$$\rho\frac{Du}{Dt}=\rho f_x+\frac{\partial(-p+\tau_{xx})}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}$$

and analogically for the remaining directions:

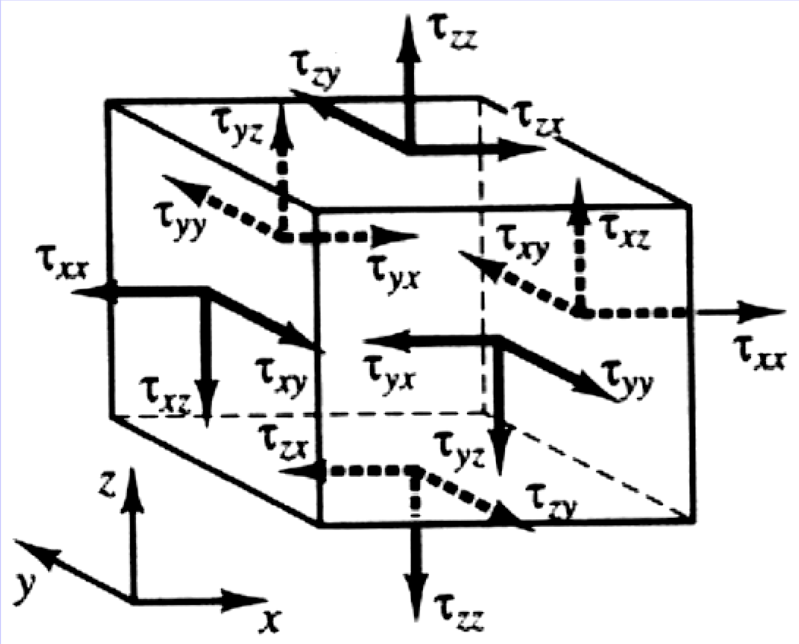
$$\rho\frac{Dv}{Dt}=\rho f_y+\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial(-p+\tau_{yy})}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}$$

$$\rho\frac{Dw}{Dt}=\rho f_z+\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial(-p+\tau_{zz})}{\partial z}$$

Three above scalar equations may be written in the form of the equivalent single vector equation:

$$\rho \frac{D\bar{u}}{Dt} = \rho\bar{f} + div[P]$$

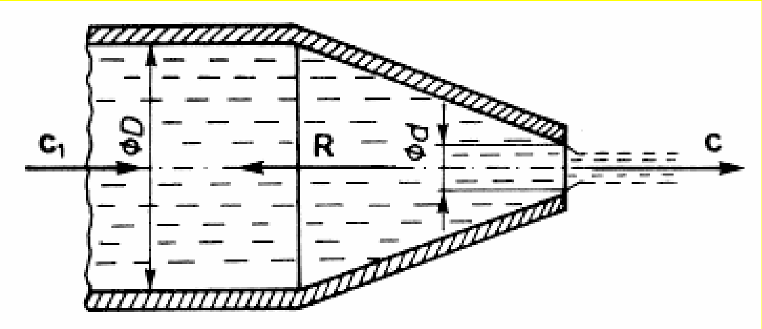
where: $[P]$ is the tensor describing the state of stress in the fluid, as shown in the picture, supplemented with pressure added to the normal stresses



The unknowns in the momentum conservation equations are: pressure, velocity components and stresses representing the surface viscous forces. Even after adding the mass conservation equation to the system, the number of unknowns is greater than the number of equations. In order to reduce the number of unknowns and close the system an appropriate fluid model must be introduced.

Examples of application of the momentum conservation principle to the solution of simple fluid mechanics problems

Example no. 1



Water is ejected with the mean velocity c=15 [m/s] from the nozzle of diameters D=80 [mm] i d=20 [mm] . Disregarding the pressure difference calculate the hydrodynamic reaction force exerted by the water stream on the duct.

The reaction R in steady motion is:

$$R = \rho \cdot Q \cdot \left(c - c_1\right)$$

The flow intensity Q and velocity c1 is calculated from the continuity equation:

$$Q = c \cdot \frac{\pi \cdot d^2}{4} = c_1 \cdot \frac{\pi \cdot D^2}{4}$$

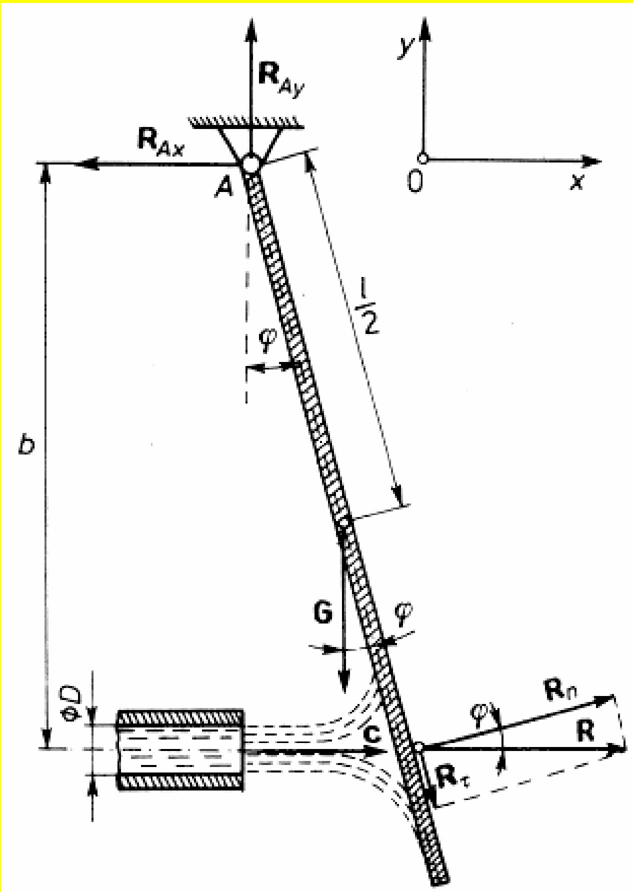
Hence we have:

$$Q = c \cdot \frac{\pi \cdot d^2}{4} \qquad c_1 = c \cdot \frac{d^2}{D^2} \qquad R = \rho \cdot c^2 \cdot \frac{\pi \cdot d^2}{4} \cdot \left(1 - \frac{d^2}{D^2}\right)$$

After substituting the numerical values we obtain:

$$R = 1000 \cdot 15^2 \cdot \frac{\pi \cdot 0.02^2}{4} \cdot \left(1 - \frac{0.02^2}{0.08^2}\right) = 66.25[N]$$

Example no. 2



The stream of perfect liquid of density ρ flows out of the nozzle and hits the ideally smooth plate of weight G and length l. The plate can rotate around the bearing A located at the distance b from the nozzle axis. Knowing that the intensity of the outflowing stream is Q, and the nozzle diameter is D, calculate the components of the reaction in the bearing and the angle φ of inclination of the plate in the state of equilibrium.

The hydrodynamic force R is resolved into the components normal and tangential to the plate:

$$\vec{R} = \vec{R}_n + \vec{R}_\tau$$

In the perfect liquid the tangential component is equal zero, consequently the entire reaction is represented by the normal component:

$$R_n = R \cdot \cos\varphi$$

Then we have:

$$R = \rho \cdot c \cdot Q$$

$$c = \frac{4 \cdot Q}{\pi \cdot D^2} \qquad R_n = \frac{4 \cdot \rho \cdot Q^2}{\pi \cdot D^2} \cdot \cos\varphi$$

The components of the reaction in the bearing are calculated from the equations of forces projections onto the axes x and y:

$$\sum P_{ix} = R_n \cdot \cos\varphi - R_{Ax} = 0$$

$$\sum P_{iy} = R_{Ay} - G - R_n \cdot \sin\varphi = 0$$

Hence we obtain:

$$R_{Ax} = \frac{4 \cdot \rho \cdot Q^2}{\pi \cdot D^2} \cdot \cos^2\varphi$$

$$R_{Ay} = G + \frac{4 \cdot \rho \cdot Q^2}{\pi \cdot D^2} \cdot \cos\varphi \cdot \sin\varphi = G + \frac{2 \cdot \rho \cdot Q^2}{\pi \cdot D^2} \cdot \sin 2\varphi$$

The inclination angle of the plate in the state of equilibrium is calculated from the equation of moments with respect to A:

$$\sum M_A = R_n \cdot \frac{b}{\cos\varphi} - G \cdot \frac{l}{2} \cdot \sin\varphi = 0$$

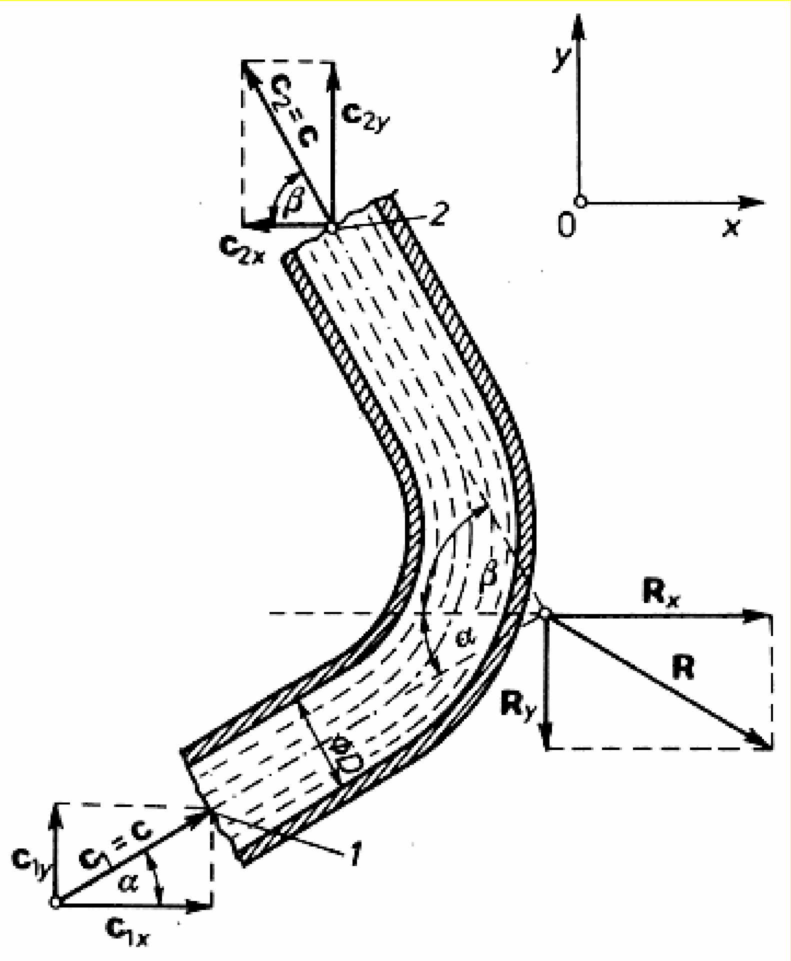
We obtain:

$$\sin\varphi = \frac{2 \cdot R_n \cdot b}{G \cdot l \cdot \cos\varphi}$$

After substituting the formula for the reaction finally we get:

$$\varphi = \arcsin\frac{8 \cdot \rho \cdot Q^2 \cdot b}{\pi \cdot G \cdot l \cdot D^2}$$

Example no. 3



Water flows through the bent pipe of diameter D=80 [mm] with flow intensity Q=0.08 [m3/s]. Disregarding the losses, calculate the force exerted by the water stream on the pipe. The inlet part of the pipe is located at the angle α=π/6 to the horizon, while the outlet part is located at the angle π/3. In both inlet and outlet cross-sections of the pipe the pressure is equal to the ambient pressure p_b.**

The components of the hydrodynamic force are respectively:

$$R_x = \rho \cdot Q \cdot \left(c_{1x} - c_{2x}\right)$$

$$R_y = \rho \cdot Q \cdot \left(c_{1y} - c_{2y}\right)$$

Where:

$$c_{1x} = c \cdot \cos\alpha \qquad c_{2x} = -c \cdot \cos\beta$$

$$c_{1y} = c \cdot \sin\alpha \qquad c_{2y} = c \cdot \sin\beta$$

What gives:

$$R_x = \rho \cdot Q \cdot c \cdot \left(\cos\alpha + \cos\beta\right)$$

$$R_y = \rho \cdot Q \cdot c \cdot \left(\sin\alpha - \sin\beta\right)$$

After substitution: $$c = \frac{4 \cdot Q}{\pi \cdot D^2}$$

We obtain:

$$R_x = \frac{4 \cdot \rho \cdot Q^2}{\pi \cdot D^2} \cdot (\cos\alpha + \cos\beta)$$

$$R_y = \frac{4 \cdot \rho \cdot Q^2}{\pi \cdot D^2} \cdot (\sin\alpha - \sin\beta)$$

The resultant force is:

$$R = \sqrt{R_x^2 + R_y^2} = \frac{4 \cdot \rho \cdot Q^2}{\pi \cdot D^2} \cdot \sqrt{2 \cdot [1 + \cos(\alpha + \beta)]}$$

The sum of angles is: $$\alpha + \beta = \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}$$

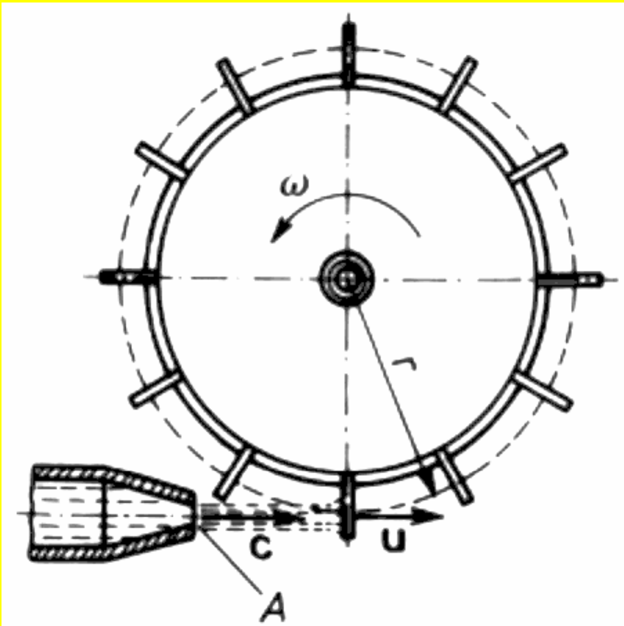
Hence we have:

$$R = \frac{4 \cdot \sqrt{2} \cdot \rho \cdot Q^2}{\pi \cdot D^2}$$

After substituting the numerical values we obtain:

$$R = \frac{4 \cdot \sqrt{2} \cdot 1000 \cdot 0.08^2}{3.1415 \cdot 0.08^2} = 1802[N]$$

Example no. 4



The stream of water of intensity q=0.01 [m3/s] flows out of the nozzle and hits the flat blades of the water wheel with the mean radius r=1.0 [m]. Disregarding the losses, calculate the effective power and the efficiency of the wheel, if its angular velocity is equal to ω=5.0 [1/s], and the cross-section area of the duct is equal to A=500 [mm**2]. For what angular velocity of the wheel the effective power reaches the highest value?**

The effective power of the wheel is determined by the relation:

$$N_u = M \cdot \omega$$

Where the moment M results from the moment of momentum principle

$$M = \rho \cdot Q \cdot (c - u) \cdot r$$

Hence: $$N_u = \rho \cdot Q \cdot (c - u) \cdot \omega \cdot r$$

Where we have: $$u = \omega \cdot r \qquad c = {Q}/{A}$$

What gives: $$N_u = \rho \cdot Q \cdot \left(\frac{Q}{A} - \omega \cdot r \right) \cdot \omega \cdot r$$

After subsitution of the numerical values we get:

$$N_u = 1000 \cdot 0.01 \cdot \left(\frac{0.01}{0.0005} - 5 \cdot 1 \right) \cdot 5 \cdot 1 = 750[W]$$

In turn the power delivered to the wheel is described by the formula:

$$N_d = \rho \cdot g \cdot Q \cdot H$$

Where the water head H is equal to:

$$H = \frac{c^2}{2 \cdot g}$$

Moreover: $c = Q / A$

What gives:

$$N_d = \frac{\rho \cdot Q^3}{2 \cdot A^2} = \frac{1000 \cdot 0.01^3}{2 \cdot 0.0005^2} = 2000[W]$$

The efficiency of the wheel is then equal to:

$$\eta = \frac{N_u}{N_d} = \frac{750}{2000} = 0.375$$

In order to determine the angular velocity corresponding to the maximum power, the equation for effective power must be re-formulated and differentiated with respect to the angular velocity

$$N_u = \rho \cdot c \cdot A \cdot \left(c - \omega \cdot r\right) \cdot \omega \cdot r = \rho \cdot A \cdot r \cdot \left(c^2 \cdot \omega - c \cdot \omega^2 \cdot r\right)$$

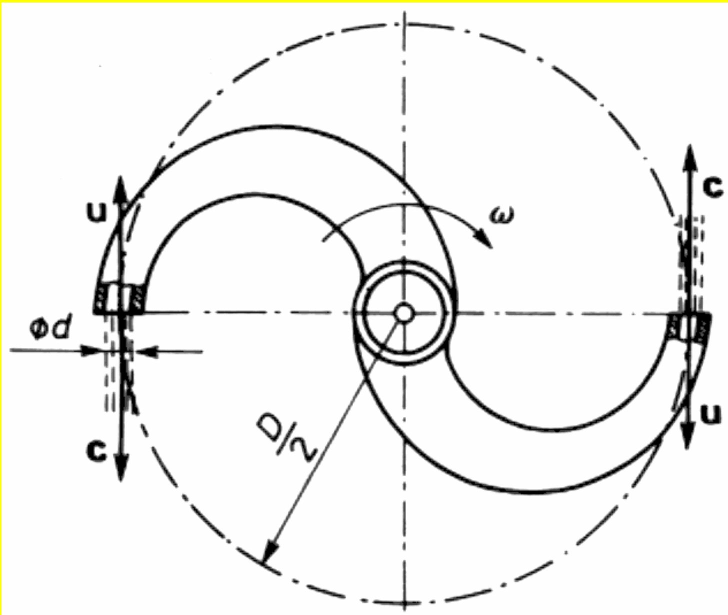
$$\frac{\partial N_u}{\partial \omega} = \rho \cdot A \cdot r \cdot \left(c^2 - 2 \cdot c \cdot \omega \cdot r\right) = 0$$

The condition for extremum

After subsituting the numerical values we obtain:

$$\omega = \frac{c}{2 \cdot r} = \frac{Q}{2 \cdot r \cdot A} = \frac{0.01}{2 \cdot 1 \cdot 0.0005} = 10\left[{}^{1}\!/_{s}\right]$$

Example no. 5



The Segner wheel of diameter D is supplied with water having the flow intensity Q. Disregarding the frictional losses and flow losses determine the angular velocity of the wheel rotation ω. Assume the outlet diameter of the nozzles equal to d. Assume that the resultant moment on the wheel is zero.

The Segner wheel rotates in the direction opposite to the water outflow, hence the absolute outflow velocity c is:

$$c = w - u$$

Where: $u = \omega \cdot \dfrac{D}{2}$ $\qquad w = \dfrac{0.5 \cdot Q \cdot 4}{\pi \cdot d^2} = \dfrac{2 \cdot Q}{\pi \cdot d^2}$

The moment of the hydrodynamic reaction based on the moment of momentyum pronciple is:

$$M = \rho \cdot Q \cdot \frac{D}{2} \cdot c = \rho \cdot Q \cdot \frac{D}{2} \cdot (w - u)$$

As we disregard the frictional losses there must be M=0, what leads to:

$$w - u = 0 \rightarrow w = u$$

After subsituting the relations for the velocities w and u we obtain:

$$\frac{2 \cdot Q}{\pi \cdot d^2} = \omega \cdot \frac{D}{2} \rightarrow \omega = \frac{4 \cdot Q}{\pi \cdot d^2 \cdot D}$$